Integration Theory — Problem Farm 1.1
MAT 725, Fall 2024 — D. Ivanšić

Bounded Variation

Type A problems (5pts each)

A1. Show that if f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$.

A2. Let $f, g : [a, b] \to \mathbf{R}$ be of bounded variation. Show that for any $c \in \mathbf{R}$ the functions $f \pm g$, cf are of bounded variation.

A3. Let $f(x) = \sin \frac{1}{x}$ if $x \in (0, 1]$, $f(0) = 0$. Is f of bounded variation on $[0, 1]$?

A4. Let $f : [a, b] \to \mathbf{R}$ be a function, and suppose there is a partition $(c_0, c_1, \ldots, c_n)$ of $[a, b]$ such that on every subinterval $[c_{i-1}, c_i]$ the function f is monotone. Determine the variation of f over $[a, b]$.

A5. Let $f, g : [a, b] \to \mathbf{R}$ be of bounded variation. In **A2** one shows that $f + g$ is of bounded variation and that $V(f + g) \leq V(f) + V(g)$. Give a simple example of f and g so that $V(f + g) < V(f) + V(g)$.

Type B problems (8pts each)

B1. Let $f, g : [a, b] \to \mathbf{R}$ be of bounded variation. Show that $f \cdot g$ is of bounded variation. Additionally, if there exists an $\epsilon > 0$ such that $g(x) \geq \epsilon$ on $[a, b]$, show that $\frac{f}{g}$ is of bounded variation.

B2. Let $f(x) = x^2 \sin \frac{1}{x}$ if $x \in (0, 1]$, $f(0) = 0$. Show that f is of bounded variation on $[0, 1]$. Why can't you use Corollary 1.10? (Use the Mean Value Theorem in $V_{\mathcal{P}}$ instead.)

B3. Let $f_n : [a, b] \to \mathbf{R}$ be a sequence of functions of bounded variation so that $f_n \to f$ (pointwise). If V_n is the variation of f_n, and $V_n \leq M < \infty$ for all $n \in \mathbf{N}$, show that f is a function of bounded variation and that $V \leq M$.

B4. Give an example of a uniformly convergent sequence f_n of functions of bounded variation whose limit is not of bounded variation.

B5. Let $f : [a, b] \to \mathbf{R}$ be a function such that for every $\delta > 0$, $f : [a + \delta, b] \to \mathbf{R}$ is of bounded variation. Assume further that $V_{[a+\delta,b]} \leq M < \infty$ for every $\delta > 0$. Show that $f : [a, b] \to \mathbf{R}$ is of bounded variation. Give a (super-simple) counterexample to show that $V_{[a,b]} \leq M$ does not necessarily follow. What additional condition will guarantee $V_{[a,b]} \leq M$?

B6. Let $f : [a, b] \to \mathbf{R}$ be a continuous function of bounded variation. Then, like in the proof of Jordan's Theorem, we define the function $V(x) = V_{[a,x]}$. Show that $V(x)$ is continuous, and that this implies that $P(x)$ and $N(x)$ are continuous, too. Hints: use additivity of V over intervals to get continuity of $V(x)$. Show first this tool for estimating $V_{[c,x]}$: if $c \in [a, b]$, $x > c$, and $\mathcal{P} = (a, x_1, \ldots, x_{n-2}, c, x)$ is a partition of $[a, x]$ that includes c, then

$$V_{\mathcal{P}} - |f(x) - f(c)| \leq V_{[a,x]} - V_{[c,x]}.$$

Type C problems (12pts each)

C1. Is Thomae's function (5.1.6.h in Bartle & Sherbert) of bounded variation on [0,1]?

Integration Theory — Problem Farm 1.2 **MAT 725, Fall 2024 — D. Ivanšić**

Rectifiable Curves

Type A problems (5pts each)

A1. Let C be the line segment $\mathbf{r} : [0,1] \to \mathbf{R}^3$: $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$. Find $L(C)$ from the definition.

A2. Let C be the curve $\mathbf{r} : [a,b] \to \mathbf{R}^3$: $\mathbf{r}(t) = \mathbf{r}_0$, if $t \in [a,c]$ and $\mathbf{r}(t) = \mathbf{r}_1$, if $t \in (c,b]$ for some $c \in [a,b)$. Find $L(C)$ from the definition.

A3. Let $f : [a,b] \to \mathbf{R}$ be a function. Parametrize the graph C of f in the usual way: $\mathbf{r}(t) = (t, f(t))$, $t \in [a,b]$. Show that C is rectifiable if and only if f is of bounded variation.

A4. Show that a curve C given by $\mathbf{r} : [a,b] \to \mathbf{R}^3$ is rectifiable if and only if both curves $\mathbf{r}|_{[a,c]} : [a,c] \to \mathbf{R}^3$ and $\mathbf{r}|_{[c,b]} : [c,b] \to \mathbf{R}^3$ are rectifiable. (There is no need for writing out sums here, just use existing theorems.)

A5. For any $c \geq 0$ and d such that $c + d \geq 0$, show that $c - \sqrt{|d|} \leq \sqrt{c+d} \leq \sqrt{c} + \sqrt{|d|}$. When does equality hold?

A6. Give a simple example that shows that conclusion of problem **B2**, $L(C) = L(C_1) + L(C_2) + d(\mathbf{r}_1(c), \mathbf{r}_2(c))$, is not valid if $\mathbf{r}_2$ is not continuous at c.

A7. Give a simple example that shows that **C1** is not true if we remove the assumption of continuity for $\mathbf{r}$. That is, give an example of a simple (discontinuous) curve $\mathbf{r} : [a,b] \to \mathbf{R}^3$ so that there is an $M < L(C)$ such that for every $\delta > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ with $||\mathcal{P}|| < \delta$ and $l_{\mathcal{P}} < M$.

Type B problems (8pts each)

B1. Let C be the line segment $\mathbf{r} : [0,1] \to \mathbf{R}^3$ (see **A1**). Show $L(C) = \sqrt{V(x)^2 + V(y)^2 + V(z)^2}$, where $V(x)$ is the variation of the x-coordinate function of $\mathbf{r}$, etc. But:

a) Give a curvy 2-dimensional counterexample $\mathbf{r} : [0,1] \to \mathbf{R}^2$ that shows above is not true in general.

b) Give a counterexample $\mathbf{r} : [0,1] \to \mathbf{R}^2$ that shows above is not true even for piecewise-linear curves.

B2. Suppose that a curve C given by $\mathbf{r} : [a,b] \to \mathbf{R}^3$ is rectifiable. If C_1 and C_2 are the restrictions $\mathbf{r}|_{[a,c]} : [a,c] \to \mathbf{R}^3$ and $\mathbf{r}|_{[c,b]} : [c,b] \to \mathbf{R}^3$ (rectifiable by **A4**), show that $L(C) = L(C_1) + L(C_2)$. (See proof of Theorem 1.2.)

B3. Suppose rectifiable curves C_1 and C_2 are given by functions $\mathbf{r}_1 : [a,c] \to \mathbf{R}^3$ and $\mathbf{r}_2 : [c,b] \to \mathbf{R}^3$, where $\mathbf{r}_2$ is continuous at c. Define $\mathbf{r} : [a,b] \to \mathbf{R}^3$ as $\mathbf{r}(t) = r_1(t)$, if $t \in [a,c]$, and $\mathbf{r}(t) = r_2(t)$, if $t \in (c,b]$. Show that $L(C) = L(C_1) + L(C_2) + d(\mathbf{r}_1(c), \mathbf{r}_2(c))$, where d is distance between points in $\mathbf{R}^3$.

B4. Prove the theorem at the end of section 1.2: if C is the curve $\mathbf{r} : [a, b] \to \mathbf{R}^3$, $\mathbf{r}(t) = (x(t), y(t), z(t))$, and x, y, z all have continuous derivatives on $[a, b]$, then

$$L(C) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Start with the sum $l_{\mathcal{P}}$ and use the Mean Value Theorem on $x(t_i) - x(t_{i-1})$, etc. Note that it will give you different points u_i, v_i, w_i in the interval $[t_{i-1}, t_i]$ for each of the x, y and z components. Now use **A5** and uniform continuity of x', y' and z' to show this expression can be made close to one where $u_i = v_i = w_i$, which is a Riemann sum for the function $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

Type C problems (12pts each)

C1. Let C be a continuous curve $\mathbf{r} : [a, b] \to \mathbf{R}^3$. Show that $L(C) = \lim_{||\mathcal{P}|| \to 0} l_{\mathcal{P}}$, that is, show that for every $M < L(C)$ there exists a $\delta > 0$, such that if $||\mathcal{P}|| < \delta$, then $l_{\mathcal{P}} > M$. (See the proof of 1.9.)

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Riemann-Stieltjes Integral

Type A problems (5pts each)

A1. Use the definition to find $\int_a^b f\,d\varphi$ in the following cases:
a) φ is a constant function, b) f is constant.

A2. Compute $\int_0^{\frac{\pi}{2}} x^2\,d\sin x$ using **B2**.

A3. Show Cauchy's criterion: f is Riemann-Stieltjes integrable if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any two tagged partitions $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ with $||\dot{\mathcal{P}}||$, $||\dot{\mathcal{Q}}|| < \delta$ we have $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \epsilon$.

A4. Give an example (simple — **A1** can help!) where $a < c < b$ and $\int_a^c f\,d\varphi$ and $\int_c^b f\,d\varphi$ both exist, but $\int_a^b f\,d\varphi$ does not. Does this contradict Theorem 1.17?

A5. Prove Theorem 1.16a: If $\int_a^b f\,d\varphi$ exists, so do $\int_a^b cf\,d\varphi$ and $\int_a^b f\,d(c\varphi)$ and $\int_a^b cf\,d\varphi = c\int_a^b f\,d\varphi = \int_a^b f\,d(c\varphi)$.

A6. Prove Theorem 1.16c: If $\int_a^b f\,d\varphi$ and $\int_a^b f\,d\psi$ exist, then $\int_a^b f\,d(\varphi+\psi)$ exists and $\int_a^b f\,d(\varphi+\psi) = \int_a^b f\,d\varphi + \int_a^b f\,d\psi$.

A7. Prove the Mean Value Theorem: If f is continuous and φ is increasing on $[a, b]$, then there exists a $c \in [a, b]$ such that $\int_a^b f\,d\varphi = f(c)(\varphi(b) - \varphi(a))$.

Type B problems (8pts each)

B1. Let $\varphi : [a, b] \to \mathbf{R}$ be a step function with subdivision $a = a_0 < a_1 < \cdots < a_n = b$ of $[a, b]$ such that $\varphi|_{(a_{i-1},a_i)}$ is constant. Set $\varphi(a_i-) = \lim_{x\to a_i-} \varphi(x)$ and $\varphi(a_i+) = \lim_{x\to a_i+} \varphi(x)$ ($\varphi(a_0-) = \varphi(a)$, $\varphi(a_m+) = \varphi(b)$). Show that $\int_a^b f\,d\varphi = \sum_{i=0}^n f(a_i)(\varphi(a_i+) - \varphi(a_i-))$ for a continuous $f : [a, b] \to \mathbf{R}$. Hint: use induction on n, applying Theorem 1.17.

B2. If f and φ' are both continuous, prove that $\int_a^b f\,d\varphi = \int_a^b f\varphi'$, where the latter is a Riemann integral.

B3. Prove Theorem 1.17: $\int_a^b f\,d\varphi$ exists, and $c \in (a, b)$, then $\int_a^c f\,d\varphi$ and $\int_c^b f\,d\varphi$ both exist, and $\int_a^b f\,d\varphi = \int_a^c f\,d\varphi + \int_c^b f\,d\varphi$. (See proof of corresponding theorem for Riemann integrals, 7.2.9.)

Type C problems (12pts each)

C1. Suppose f is continuous and φ is of bounded variation on $[a, b]$. Show:
a) $\psi(x) = \int_a^x f\, d\varphi$ is of bounded variation on $[a, b]$.
b) If g is continuous on $[a, b]$, then $\int_a^b g\, d\psi = \int_a^b gf\, d\varphi$.

C2. Suppose f is continuous and φ and ψ are of bounded variation on $[a, b]$. Show that $\int_a^b f\, d(\varphi\psi) = \int_a^b f\psi\, d\varphi + \int_a^b f\varphi\, d\psi$.

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Open and Closed Sets

Type A problems (5pts each)

A1. Prove Proposition 12 in 1.4.

A2. Let $A = \{1 + (-1)^n \frac{1}{n} \mid n \in \mathbf{N}\}$. Determine $\overline{A}$ with explanation.

A3. Let $A = \mathbf{Q}^c \cap [0, 1]$. Determine $\overline{A}$ with explanation.

A4. Determine $\operatorname{Int} \mathbf{Q}$ with explanation.

A5. Show that a finite subset of $\mathbf{R}$ is always closed.

A6. Is $A = \{\frac{1}{n} \mid n \in \mathbf{N}\}$ compact? Justify your answer.

Type B problems (8pts each)

B1. Prove Proposition 13 in 1.4.

B2. For a set $A \subseteq \mathbf{R}$, show that $x \in \overline{A}$ if and only if there exists a sequence (x_n) such that $x_n \in A$ for all $n \in \mathbf{N}$ and $x_n \to x$. Conclude that A is closed if and only if every convergent sequence in A converges to an element of A.

B3. For a set $A \subseteq \mathbf{R}$, show that $\overline{A} = \cap_{A \subset F,\ F \text{ closed}} F$. Conclude that $\overline{A}$ is the smallest closed set that contains A in the sense that if F is closed and $A \subseteq F$, then $\overline{A} \subseteq F$.

B4. For a set $A \subseteq \mathbf{R}$, show that $\operatorname{Int} A = \cup_{U \subset A,\ U \text{ open}} U$. Conclude that $\operatorname{Int} A$ is the largest open set contained in A in the sense that if U is open and $U \subseteq A$, then $U \subseteq \operatorname{Int} A$.

B5. Show that a set $A \subset \mathbf{R}$ is compact if and only if every sequence in A has a subsequence that converges to an element of A. (Slap Borel's Heine.)

B6. Let $f : \mathbf{R} \to \mathbf{R}$. Show that f is continuous if and only if for every open set $V \subseteq \mathbf{R}$, $f^{-1}(V)$ is an open set.

B7. For a set $A \subseteq \mathbf{R}$, show that $\operatorname{Int}(A^c) = (\overline{A})^c$.

B8. For sets $A, B \subseteq \mathbf{R}$, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Type C problems (12pts each)

(none)