Integration Theory — Lecture notes
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8.1 Pointwise and Uniform Convergence

Definition. Let $A \subseteq \mathbf{R}$ and let $f_n : A \to \mathbf{R}$ be a function for every $n \in \mathbf{N}$. We say that (f_n) is a *sequence of functions.* For each $x \in A$, (f_n) gives rise to a sequence of numbers $(f_n(x))$. These sequences may converge for some x and diverge for others.

Definition 8.1.1. Let $(f_n) : A \to \mathbf{R}$ be a sequence of functions, $A_0 \subseteq A$ and let $f : A_0 \to \mathbf{R}$. We say that (f_n) *converges to* f *on* A_0 if for every $x \in A_0$, $f_n(x) \to f(x)$. In this case, we say that f *is the limit of* (f_n) *on* A_0, or that (f_n) *converges pointwise on* A_0.

Notation. $\lim f_n = f$ on A_0 $\qquad$ $f_n \to f$ on A_0

For the following examples, find the set A_0 on which the sequence (f_n) converges pointwise, and find the limit of (f_n) on A_0. Draw graphs of the functions f_n to help you see what is happening.

Example. $f_n(x) = \dfrac{x}{n}$

Example. $f_n(x) = x^n$

Example. $f_n(x) = \dfrac{x^2 + nx}{n}$

Note. $f_n \to f$ on A_0 if and only if for every $x \in A_0$ and every $\varepsilon > 0$ there is a $K \in N$ such that for all $n \geq K$, $|f_n(x) - f(x)| < \varepsilon$. Note that K depends on x and ε.

Compare the above statement to the definition below.

Definition 8.1.4. A sequence $(f_n) : A \to \mathbf{R}$ of functions *converges to* $f : A_0 \to \mathbf{R}$ *uniformly on* $A_0 \subseteq A$ if for every $\varepsilon > 0$ there is a $k \in N$ such that for all $n \geq K$, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in A_0$.
Note that K depends only on ε, but not on x any more.

Notation. $f_n \rightrightarrows f$ on A_0 $\quad$ $f_n(x) \rightrightarrows f(x)$ on A_0

"uniformly" means convergence is occuring "with equal speed" across all $x \in A_0$.

Note. Clearly, if $f_n \rightrightarrows f$ on A_0, then $f_n \to f$ on A_0.

For the following examples, determine if there is set A_0 on which the sequence (f_n) converges uniformly to a function f. and find the limit of (f_n) on A_0.

Example. $f_n(x) = \dfrac{x}{n}$

Example. Use negation of the definition of uniform continuity to show that $f_n(x) = x^n$ does not converge uniformly on $[0, 1]$.

Lemma 8.1.5. A sequence $(f_n) : A \to \mathbf{R}$ does not converge uniformly on A_0 to a function $f : A_0 \to \mathbf{R}$ if and only if there exists an $\varepsilon_0 > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence $x_k \in A_0$ such that $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$ for all $k \in \mathbf{N}$.

Definition 8.1.7. Let $A \subseteq \mathbf{R}$, and let $f : A \to \mathbf{R}$ be a bounded function (recall this means that the set $f(A)$ is bounded in $\mathbf{R}$). We define *the uniform norm of f on A* as

$$||f||_A = \sup\{f(x) \mid x \in A\}$$

Note that $||f||_A \leq \varepsilon$ if and only if $f(x) \leq \varepsilon$ for every $x \in A$.

Lemma 8.1.8. A sequence $(f_n) : A \to \mathbf{R}$ of bounded functions converges uniformly to $f : A \to \mathbf{R}$ if and only if $||f_n - f|| \to 0$.

Proof.

Example. Show that the sequence $f_n(x) = \dfrac{1}{n}\sin(nx + n)$ converges uniformly to the zero function.

Example. Show that sequence $f_n(x) = x^n$ does not converge uniformly on the interval $[0, 1]$.

Cauchy Criterion for Uniform Convergence 8.1.10. Let $(f_n) : A \to \mathbf{R}$ be a sequence of bounded functions. Then (f_n) converges uniformly to a bounded function $f : A \to \mathbf{R}$ if and only if for every $\varepsilon > 0$ there is a $K \in \mathbf{N}$ such that for all $m, n \geq K$, $||f_n - f_m|| < \varepsilon$.

Proof.

8.2 Interchange of Limits

Example. Let $f_n : [0,1] \to \mathbf{R}$, $f_n(x) = x^n$. $\lim f_n = f$, where $f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1 \end{cases}$
Comment on the continuity and differentiability of f_n and f.

Example. Let $f_n : [0,1] \to \mathbf{R}$ be the function pictured at right. Determine the following and comment.

$f = \lim f_n =$ $\qquad$ $\int_0^1 f =$

$\int_0^1 f_n =$ $\qquad$ $\lim \int_0^1 f_n =$

Theorem 8.2.2. Let $f_n : A \to \mathbf{R}$ be a a sequence of continuous functions and let f_n converge uniformly on A to a function $f : A \to \mathbf{R}$. Then f is continuous on A.

Proof.

Theorem 8.2.3. Let J be a bounded interval and let $(f_n) : J \to \mathbf{R}$ be a a sequence of functions differentiable on J. Suppose there exists an $x_0 \in J$ such that $(f_n(x_0))$ converges and that (f_n') converges uniformly on J to a function $g : J \to \mathbf{R}$. Then (f_n) converges uniformly on J to a a function $g : J \to \mathbf{R}$ that is differentiable on J and $f' = g$.

Proof.

Theorem 8.2.4. Let $(f_n) : [a, b] \to \mathbf{R}$ be a a sequence of Riemann-integrable functions (that is, $f_n \in \mathcal{R}[a, b]$) that converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbf{R}$. Then $f \in \mathcal{R}[a, b]$, and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

Proof.

Theorems 8.2.5 and 8.2.6. are some variations on convergence theorems 8.2.4 and 8.2.2 — read in book.

8.3 The Exponential and Logarithmic Functions

We wish to rigorously define the exponential function e^x.

Theorem 8.3.1. There exists a function $E : \mathbf{R} \to \mathbf{R}$ such that:

i) $E'(x) = E(x)$ for all $x \in \mathbf{R}$.
ii) $E(0) = 1$.

Proof.

Corollary 8.3.2. The function $E(x)$ has derivatives of every order and $E^{(n)}(x) = E(x)$.

Corollary 8.3.3. If $x \geq -1$, then $1 + x \leq E(x)$, with equality achieved only for $x = 0$.

Proof.

Theorem 8.3.4. The function $E : \mathbf{R} \to \mathbf{R}$ that satisfies the conditions from Theorem 8.3.1 is unique.

Proof.

Definition 8.3.5. The unique function $E : \mathbf{R} \to \mathbf{R}$ satisfying $E'(x) = E(x)$ and $E(0) = 1$ is called *the exponential function.* The number $e = E(1)$ is called the *Euler number*, and we often write $E(x) = e^x$, because (as established in the next theorem), $E(x)$ has properties of taking powers.

Theorem 8.3.6. The exponential function $E(x)$ has the following properties for all $x, y \in \mathbf{R}$:

iii) $E(x) \neq 0$.

iv) $E(x + y) = E(x)E(y)$.

v) $E(r) = e^r$, for all $r \in \mathbf{Q}$.

Proof.

Theorem 8.3.7. The exponential function $E(x)$ is strictly increasing and has range $(0, \infty)$. Furthermore,

vi) $\lim_{x\to\infty} E(x) = \infty$ and $\lim_{x\to-\infty} E(x) = 0$

Proof.

Note.

— Property (v) says that for rational numbers $\frac{m}{n}$, $E(\frac{m}{n})$ is the same as taking the power $e^{\frac{m}{n}} = \sqrt[n]{e^m}$, which we have defined before. Because E is continuous, $E(x)$ thus continuously extends the idea of rational powers of e to irrational powers without (explicitly) resorting to limits.

— Property (iv) tells us that the function $E : (\mathbf{R}, +) \to (\mathbf{R}^+, \cdot)$ is a group isomorphism. In other words, addition of real numbers and multiplication of positive numbers is essentially the same thing, which is what inventors of logarithmic tables realized and used to simplify computation of products and quotients.

Definition 8.3.8. Since $E : \mathbf{R} \to (0, \infty)$ is increasing and differentiable, and $E'(x) \neq 0$ it follows from Theorem 6.1.8. that E has an inverse function $L : (0, \infty) \to \mathbf{R}$, called the *(natural) logarithm*, also denoted ln. Then we have the standard equations for inverse functions:

$$(L \circ E)(x) = x \Longleftrightarrow \ln e^x = x \qquad \text{and} \qquad (E \circ L)(y) = y \Longleftrightarrow e^{\ln y} = y$$

Theorem 8.3.9. The logarithm $L : (0, \infty) \to \mathbf{R}$ is strictly increasing and has the following properties. For all $x, y > 0$:

vii) $L'(x) = \frac{1}{x}$.

viii) $L(xy) = L(x) + L(y)$.

ix) $L(1) = 0$, $L(e) = 1$.

x) $L(x^r) = rL(x)$ for $r \in \mathbf{Q}$.

xi) $\lim_{x\to\infty} L(x) = \infty$ and $\lim_{x\to 0} L(x) = -\infty$.

Proof.

Definition 8.3.10. Now we can extend the definition of the power function to any exponent $\alpha \in \mathbf{R}$, including irrationals, by setting

$$x^\alpha = E(\alpha \ln x) = e^{\alpha \ln x}$$

Note. If $\alpha = r \in \mathbf{Q}$, we have

$$x^\alpha = E(\alpha L(x)) = E(rL(x)) = E(L(x^r)) = x^r$$

which agrees with the original definition of the power function for rational powers.

Theorems 8.3.11–13. develop the usual properties of powers for the extended definition (read).

8.4 Trigonometric Functions

We establish existence of functions $\sin x$, $\cos x$.

Theorem 8.4.1. There exist functions $C, S : \mathbf{R} \to \mathbf{R}$ such that

i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbf{R}$.

ii) $C(0) = 1$, $C'(0) = 0$, $S(0) = 0$, $S'(0) = 1$.

Proof.

Corollary 8.4.2. The functions C and S from Theorem 8.4.1 satisfy:

iii) $C'(x) = -S(x)$ and $S'(x) = C(x)$.

Corollary 8.4.3. The functions C and S satisfy the Pythagorean identity:

iv) $C(x)^2 + S(x)^2 = 1$.

Proof.

Theorem 8.4.4. The functions C and S from Theorem 8.4.1 are unique.

Proof.

Definition 8.4.5. Since the functions C and S are unique, we give them names:

$$C(x) = \cos x \qquad S(x) = \sin x$$

Theorem 8.4.6. If a function $f : \mathbf{R} \to \mathbf{R}$ satisfies $f''(x) = -f(x)$ then there exist $\alpha, \beta \in \mathbf{R}$ such that

$$f(x) = \alpha C(x) + \beta S(x).$$

Proof.

Theorem 8.4.7. The functions C and S have the following properties for all $x, y \in \mathbf{R}$.

v) C is even: $C(-x) = C(x)$ and S is odd: $S(-x) = -S(x)$.

vi) $C(x+y) = C(x)C(y) - S(x)S(y)$ and $S(x+y) = S(x)C(y) + C(x)S(y)$.

Proof.

Theorems 8.4.8–11. have some more usual properties of $C(x)$ and $S(x)$ — including the definition of π as the smallest positive zero of S — read.