2.1 Introduction to Measure Theory

We wish to define a function (measure)

$$m : \{\text{some subsets of } \mathbf{R}\} \to [0, \infty]$$

which captures the idea of "size," which in $\mathbf{R}$ is "length." (If we were working in $\mathbf{R}^2$ or $\mathbf{R}^3$, "size" would be "area" or "volume.")

Definition. Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbf{R}$ that contains all intervals. A function $m : \mathcal{A} \to [0, \infty] = [0, \infty) \cup \{\infty\}$ is called a *Lebesgue measure* if it possesses these properties:

1) *Meausure of an interval is its length.* If I is an interval — (a, b), $[a, b)$, $(a, b]$, $[a, b]$, where open bounds could be ∞ — then $m(I)$ =length of I (possibly ∞)

2) *Measure is translation-invariant.* If $E \in \mathcal{A}$, then for every $y \in \mathbf{R}$, $E + y = \{e + y \mid e \in E\}$ is also in $\mathcal{A}$ and $m(E + y) = m(E)$.

3) *Measure is countably additive over countable disjoint unions.* If $\{E_k, k \in \mathbf{N}\}$ is a disjoint collection of sets in $\mathcal{A}$, then $m\left(\bigcup_{k\in\mathbf{N}} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$ (disjoint collection means for every $i \neq j$, $E_i \cap E_j = \emptyset$).

It turns out, it is not possible to achieve this for $\mathcal{A} = \mathcal{P}(\mathbf{R})$ = all subsets of $\mathbf{R}$, but it is for a smaller collection, a σ-algebra called *Lebesgue measurable* sets, which contain the Borel sets.

To prove existence of such a measure function, we start with a function called *outer measure*.

2.2 Lebesgue Outer Measure

Definition. Let I be an open interval, $I=(a,b)$, where $a\in\{-\infty\}\cup\mathbf{R}$ and $b\in\mathbf{R}\cup\{\infty\}$. The *length of* I, $\ell(I)$, is defined as:

$$\ell(I)=\begin{cases} b-a, & \text{if } a,b\in\mathbf{R}\\ \infty, & \text{if } a=-\infty \text{ or } b=\infty\end{cases}$$

Definition. Let $A\subseteq\mathbf{R}$. The *outer measure of* A, $m^*(A)$ or m^*A, is defined as

$$m^*A=\inf\left\{\sum_{k=1}^{\infty}\ell(I_k)\mid A\subseteq\bigcup_{k=1}^{\infty}I_k,\quad \begin{array}{l}\text{where } \{I_k, k\in\mathbf{N}\} \text{ is}\\ \text{a cover of } A \text{ by open intervals}\end{array}\right\}$$

Note. 1) $m^*\emptyset=0$

2) If $A\subseteq B$, then $m^*A\leq m^*B$.

Example. If A is countable, then $m^*A=0$.

Proposition 2.1. If I is an interval, then $m^*I=\ell(I)$.

Proof.

Proposition 2.2. Outer measure is translation-invariant, that is, for every $A \subseteq \mathbf{R}$, $y \in \mathbf{R}$, $m^*(A+y) = m^*A$.

Proof.

Theorem 2.3. Outer measure is countably subadditive, that is, for every countable collection $\{E_k, k \in \mathbf{N}\}$ of subsets of $\mathbf{R}$, $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^* E_k$.

Proof.

2.3 Lebesgue Measurable Sets

For an outer measure m^*, we know that $m^*(A \cup B) \leq m^*A + m^*B$ holds. However, there exist disjoint sets for which

$$m^*(A \cup B) < m^*A + m^*B, \text{ which is not desirable for a measure.}$$

Setting $E = A$, $C = A \cup B$, this can be rewritten as

$$m^*(C) < m^*(E \cap C) + m^*(E^c \cap C), \text{ again, not desirable for a measure.}$$

Definition. A set E is *measurable* if for any set A

$$m^*A = m^*(A \cap E) + m^*(A \cap E^c)$$

It immediately follows that if one of A, B is measurable and A, B are disjoint, then $m^*(A \cup B) = m^*A + m^*B$.

Note.

1) Since $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$, to show E is measurable we only need to show the opposite inequality: $m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$.
2) E is measurable if and only if E^c is measurable.
3) $\emptyset$ and $\mathbf{R}$ are measurable.

Proposition 2.4. Any set of outer measure zero is measurable. In particular, all countable sets are measurable.

Proof.

Proposition 2.5. The union of a finite collection of sets is measurable.

Proof.

Proposition 2.5 shows that the collection of measurable sets is an algebra (defined like a σ-algebra, except with closure with respect to finite unions instead of countable).

Proposition 2.6. Let $E_1, \ldots, E_n$ be disjoint measurable sets and $A \subseteq \mathbf{R}$. Then

$$m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} m^*(A \cap E_k) \quad \text{and} \quad m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^* E_k$$

Proof.

Proposition 2.7. The union of a countable collection of measurable sets is measurable. The collection of measurable sets is a σ-algebra.

Proof.

Proposition 2.8. Every interval is measurable.

Proof.

Note.

1) The collection of measurable sets is a σ-algebra.
2) Every open set is measurable — it is a countable union of open intervals.
3) Every closed set is measurable — it is a complement of an open set.
4) Every F_σ and G_δ set is measurable — they are intersections and unions of countable collections of closed and open sets
5) Every Borel set is measurable — it is in the smallest σ-algebra that contains open sets and measurable sets are one σ-algebra that contains open sets.

Thus we have proved:

Theorem 2.9. The collection of measurable sets is a σ-algebra that contains the Borel sets.

Proposition 2.10. The translate of a measurable set is measurable.

Proof.

2.4 Outer/Inner Approx. of Lebesgue Measurable Sets

Let A be measurable, $m^*A < \infty$. Then for any set $B \supseteq A$ we have

$$m^*(B - A) = m^*B - m * A \qquad \text{the excision property}$$

Theorem 2.11. Let E be any set. Then measurability of E is equivalent to any of the following four conditions.

Outer approximation by open and G_δ sets:

1) For every $\varepsilon > 0$ there is an open set $U \supseteq E$ such that $m^*(U - E) < \varepsilon$.
2) There exists a G_δ-set $G \supseteq E$ such that $m^*(G - E) = 0$.

Inner approximation by closed and F_σ sets:

3) For every $\varepsilon > 0$ there is a closed set $F \subseteq E$ such that $m^*(E - F) < \varepsilon$.
4) There exists an F_σ-set $F \subseteq E$ such that $m^*(E - F) = 0$.

Proof.

Note. The theorem implies that measurable sets have form $E = G - Y = F \cup Z$ where G is a G_δ-set, F is an F_σ-set and Y and Z are sets of measure zero.

Note. For any set E there is an open set $U = \bigcup_{k=1}^{\infty} I_k$ such that $m^*U < m^*E + \varepsilon$, so assuming m^*E is finite, $m^*U - m^*E < \varepsilon$, but this does not mean that $m^*(U - E) < \varepsilon$ because $m^*(U - E) = m^*U - m^*E$ is valid only for measurable sets E.

Theorem 2.12. Let E be measurable and $m^*E < \infty$. Then for every $\varepsilon > 0$ there is a disjoint collection of open intervals $I_1, \ldots, I_n$ such that

$$m^*(E - U) + m^*(U - E) < \varepsilon, \text{ where } U = I_1 \cup \cdots \cup I_n.$$

Proof.

2.5 Countable Additivity, Continuity, Borel-Cantelli

Definition. Let $\mathcal{M}$ be the σ-algebra of measurable subsets of $\mathbf{R}$.
The function $m : \mathcal{M} \to [0, \infty]$ defined by $mE = m^*E$ is called *the Lebesgue measure.*

Theorem 2.13. Lebesgue measure is countably additive, that is, if $\{E_k, k \in \mathbf{N}\}$ is a disjoint collection of measurable sets, then

$$\bigcup_{k=1}^{\infty} E_k \text{ is measurable and } m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} mE_k$$

Proof.

Theorem 2.14. The function $m : \mathcal{M} \to [0, \infty]$ is a Lebesgue measure as defined in 2.1 (assigns length to any interval, is translation invariant and countably additive).

Theorem 2.15 (continuity of measure).

1) If $\{A_k, k \in \mathbf{N}\}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} mA_k$$

2) If $\{B_k, k \in \mathbf{N}\}$ is a descending collection of measurable sets, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} mB_k$$

Proof.

Definition. If E is measurable, we say *a property $\mathcal{P}$ holds almost everywhere on E (a.e. on E, holds for almost all $x \in E$)* if there is a subset $E_0 \subseteq E$ such that $mE_0 = 0$ and $\mathcal{P}$ holds for all $x \in E - E_0$.

The Borel Cantelli Lemma. Let $\{E_k, k \in \mathbf{N}\}$ be a collection of measurable sets satisfying $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbf{R}$ belong to at most finitely many of the sets E_k.

Proof.