Integration Theory — Lecture notes
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1.1 Functions of Bounded Variation

Definition. Let $f : [a, b] \to \mathbf{R}$ and let $\mathcal{P} = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$. Define

The variation of f with respect to $\mathcal{P}$: $$V_{\mathcal{P}} = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

The variation of f over $[a, b]$: $$V = V_{[a,b]} = \sup\{V_{\mathcal{P}} \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

Clearly $0 \le V \le \infty$. If $V \neq \infty$ we say that f is *of bounded variation on* $[a, b]$, if $V = \infty$, we say that f is *of unbounded variation on* $[a, b]$.

Example. If $f(x) = k$, then $V_{[a,b]} = 0$ for every interval $[a, b]$.

Example. If $f(x)$ is monotone, then $V_{[a,b]} = |f(b) - f(a)|$.

Example. Determine $V_{[a,b]}$ for any $a, b \in \mathbf{R}$ if $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$

Example. For $f : [0,1] \to \mathbf{R}$ pictured at right, show $V_{[0,1]} = \infty$.

Example. Show the Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases}$ is of unbounded variation.

Example. If f is Lipschitz (there is a $C > 0$ such that $|f(x) - f(u)| \leq C|x - u|$), then it is of bounded variation. In particular, if f has a continuous first derivative on $[a, b]$, then it is Lipschitz, so of bounded variation.

Theorem 1.1.

a) If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$.

b) If f and g are of bounded variation on $[a, b]$, then so are cf, $f \pm g$, fg. Furthermore, if there is a number $\varepsilon > 0$ such that $|g(x)| > \varepsilon$ for all $x \in [a, b]$, then $\frac{f}{g}$ is of bounded variation.

Proof. Homework!

Theorem 1.2. Let $f : [a, b] \to \mathbf{R}$.

a) If $[a', b'] \subseteq [a, b]$, then $V_{[a',b']} \leq V_{[a,b]}$.

b) If $a < c < b$, then $V_{[a,b]} = V_{[a,c]} + V_{[c,b]}$.

Proof.

Definition. For an $x \in \mathbf{R}$, define $x^+ = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$ $\qquad x^- = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

The numbers $x^+, x^- \geq 0$ are called the *positive and negative parts of* x.

Clearly: $|x| = x^+ + x^- \quad x = x^+ - x^- \quad x^+ = \frac{1}{2}(|x| + x) \quad x^- = \frac{1}{2}(|x| - x)$.

Given a partition $\mathcal{P}$ of $[a, b]$, define

$$P_{\mathcal{P}} = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ \qquad N_{\mathcal{P}} = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-$$

Then $V_{\mathcal{P}} = P_{\mathcal{P}} + N_{\mathcal{P}}$ and $P_{\mathcal{P}} - N_{\mathcal{P}} = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a)$. Define

$$P = P_{[a,b]} = \sup\{P_{\mathcal{P}} \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$
$$N = N_{[a,b]} = \sup\{N_{\mathcal{P}} \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

Note: $0 \leq P, N \leq \infty$

Theorem 1.6. For a function $f : [a, b] \to \mathbf{R}$, if one of P, N, V is finite, so are the others. In this case

$$V = P + N \qquad P - N = f(b) - f(a), \text{ or}$$
$$P = \frac{1}{2}(V + f(b) - f(a)) \qquad N = \frac{1}{2}(V - f(a) + f(b))$$

Proof.

Jordan's Theorem 1.7. A function $f : [a, b] \to \mathbf{R}$ is of bounded variation if and only if it is the difference of two increasing, bounded functions on $[a, b]$.

Proof.

Theorem 1.8. Every function of bounded variation has at most a countable number of discontinuities.

Proof.

Theorem 1.9. If $f : [a, b] \to \mathbf{R}$ is continuous, then $V = \lim_{||\mathcal{P}|| \to 0} V_{\mathcal{P}}$, that is, given an $M < V$, there exists a $\delta > 0$ such $V_{\mathcal{P}} > M$ for any partition $\mathcal{P}$ with $||\mathcal{P}|| < \delta$.

Proof.

Corollary 1.10. If $f : [a, b] \to \mathbf{R}$ has a continuous derivative on $[a, b]$ then

$$V = \int_a^b |f'| \qquad P = \int_a^b (f')^+ \qquad N = \int_a^b (f')^-$$

Proof.

1.2 Rectifiable curves

A curve C in a plane or space is usually given by two or three parametric equations:

Curve in plane: $x = x(t)$, $y = y(t)$, $t \in [a, b]$

Curve in space: $x = x(t)$, $y = y(t)$, $z = z(t)$, $t \in [a, b]$

We may also view it as a vector function $\mathbf{r} : [a, b] \to \mathbf{R}^2$ or $\mathbf{R}^3$.

Definition. The *graph of* C is the set $\{\mathbf{r}(t) \mid t \in [a, b]\}$. Note that the graph may have self-intersections and need not be continuous or bounded.

Definition. Given a partition $\mathcal{P} = (t_0, t_1, \ldots, t_n)$ of $[a, b]$, set

$$l_{\mathcal{P}} = \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2} = \sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$$

Note that $l_{\mathcal{P}}$ is the sum of lengths of line segments with endpoints on C. The length of the curve C (rather, of the parametrization) is defined as

$$L = L(C) = \sup\{l_{\mathcal{P}} \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

Clearly $0 \leq L(C) \leq \infty$, and if $L(C) < \infty$, we say the curve C is *rectifiable.*

Note: If $\mathbf{r}(t)$ is not continuous, then $L(C)$ counts the gap, too.

If $\mathbf{r}(t)$ traces out the graph more than once, then $L(C)$ takes into account how many times the curve has been traversed. Thus, while two parametrizations $\mathbf{r}(t)$ and $\mathbf{s}(t)$ may have the same image set, the length computed using the two parametrizations could be different, because one parametrization may trace parts of the image set more than once.

Theorem 1.13. Let a curve C be parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, $t \in [a, b]$. Then C is rectifiable if and only if all of x, y and z are of bounded variation. Furthermore,

$$V(x), V(y), V(z) \leq L(C) \leq V(x) + V(y) + V(z)$$

Proof.

Example. Let $\mathbf{r}(t) = (f(t), f(t))$, where $f : [0, 1] \to [0, 1]$ is a function of unbounded variation. Then $L(C) = \infty$, even though the graph of $\mathbf{r}$ is a line segment.

Theorem. If C is parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, $t \in [a, b]$, and each of x, y and z has a continuous derivative, then

$$L(C) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Proof. Homework!

1.3 The Riemann-Stieltjes Integral

Definition. Let $\dot{\mathcal{P}} = (x_0, x_1, \ldots, x_n)$, $t_i \in [x_{i-1}, x_i]$ be a tagged partition of $[a, b]$, $f, \varphi : [a, b] \to \mathbf{R}$ functions. The sum

$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(\varphi(x_i) - \varphi(x_{i-1}))$$

is called the Riemann-Stieltjes sum for $\dot{\mathcal{P}}$.

If there exists a number L such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||\dot{\mathcal{P}}|| < \delta$, then $|S(f, \dot{\mathcal{P}}) - L| < \varepsilon$, we say that f *is Riemann-Stieltjes integrable with respect to* φ, and call the number L the *Riemann-Stieltjes integral of* f *with respect to* φ.

Notation. $L = \displaystyle\int_a^b f\, d\varphi = \int_a^b f(x)\, d\phi(x)$

Notes. 1) If $\int_a^b f\, d\varphi$ exists, it is unique (like 7.1.2).

2) (Cauchy's criterion) $\int_a^b f\, d\varphi$ exists if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ are tagged partitions with $||\dot{\mathcal{P}}||, ||\dot{\mathcal{Q}}|| < \delta$, then $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \varepsilon$ (like 7.2.1).

3) If $\varphi(x) = x$, then $\int_a^b f\, d\varphi = \int_a^b f$, so the Riemann integral is a special case of the Riemann-Stieltjes integral.

4) If f, φ' are continuous on $[a, b]$, then $\int_a^b f\, d\varphi = \int_a^b f\varphi'$

5) Let φ be a step function, so there is a partition $(a_0, a_1, \ldots, a_n)$ of $[a, b]$ such that $\varphi|_{(a_{i-1}, a_i)}$ is constant, and set

$$\varphi(a_i-) = \lim_{x \to a_i-} \varphi(x), \quad i = 1, \ldots, n, \qquad \varphi(a_i+) = \lim_{x \to a_i+} \varphi(x), \quad i = 0, \ldots, n-1$$

$$d_i = \varphi(a_i+) - \varphi(a_i-), \quad i = 1, \ldots, n-1, \quad d_0 = \varphi(a_0+) - \varphi(a_0), \quad d_n = \varphi(a_n) - \varphi(a_n-)$$

Then, for a continuous f

$$\int_a^b f\, d\varphi = \sum_{i=0}^{n} f(a_i) d_i$$

Proof is similar to 7.1.4b, integral of a step function.

6) If f and φ have the same point of discontinuity, then $\int_a^b f\, d\varphi$ does not exist.

Proof.

Theorem 1.16. Let $f, \varphi : [a, b] \to \mathbf{R}$ be functions, $c \in \mathbf{R}$.

a) If $\int_a^b f\, d\varphi$ exists, so do $\int_a^b cf\, d\varphi$ and $\int_a^b f\, d(c\varphi)$ and

$$\int_a^b cf\, d\varphi = c \int_a^b f\, d\varphi = \int_a^b f\, d(c\varphi)$$

b) If $\int_a^b f\, d\varphi$ and $\int_a^b g\, d\varphi$ exist, then $\int_a^b f + g\, d\varphi$ exists and

$$\int_a^b f + g\, d\varphi = \int_a^b f\, d\varphi = \int_a^b g\, d\varphi$$

c) If $\int_a^b f\, d\varphi$ and $\int_a^b f\, d\psi$ exist, then $\int_a^b f\, d(\varphi + \psi)$ exists and

$$\int_a^b f\, d(\varphi + \psi) = \int_a^b f\, d\varphi + \int_a^b f\, d\psi$$

Theorem 1.17. If $\int_a^b f\, d\varphi$ exists and $c \in (a, b)$, then

$$\int_a^b f\, d\varphi = \int_a^c f\, d\varphi + \int_c^b f\, d\varphi$$

Proofs. Are similar to those for Riemann integrals.

Theorem 1.21. If $\int_a^b f\,d\varphi$, then so does $\int_a^b \varphi\,df$ exists and (like integration by parts)

$$\int_a^b f\,d\varphi = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi\,df$$

Proof.

When does $\int_a^b f\,d\varphi$ exist? Here is a sufficient condition.

Theorem 1.24. Let $f, \varphi : [a,b] \to \mathbf{R}$ and let f be continuous and φ be of bounded variation on $[a,b]$. Then $\int_a^b f\,d\varphi$ exists and

$$\int_a^b f\,d\varphi \leq \sup\{|f(x)| \mid x \in [a,b]\} \cdot V_{[a,b]}(\varphi)$$

Proof.

Mean Value Theorem for Riemann-Stieltjes Integrals 1.27. Let $f, \varphi : [a, b] \to \mathbf{R}$, and let f be continuous and φ increasing on $[a, b]$. Then there exists a number $c \in [a, b]$ such that

$$\int_a^b f\, d\varphi = f(c)(\varphi(b) - \varphi(a)$$

1.4 Open, Closed and Borel Sets

Definition. A set $U \subseteq \mathbf{R}$ is *open* if for every $x \in U$ there exists an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq U$. (Intuitively, U is open if it is "fat" around each of its points.)

Example. Show that each of the following subsets of $\mathbf{R}$ is open: (a,b), (a,∞), $(-\infty,b)$. Show that $[a,b]$ is not open.

Proposition 8.

1) $\emptyset$ and $\mathbf{R}$ are open.
2) If $U_1,\dots,U_n$ are open, then $U_1 \cap \cdots \cap U_n$ is open.
3) If $\{U_\alpha \mid \alpha \in I\}$ is a collection of open sets, then $\bigcup_{\alpha\in I} U_\alpha$ is an open set.

Recall that $\bigcup_{\alpha\in I} U_\alpha = \{x \mid x \in U_\beta \text{ for some } \beta \in I\}$ $\quad \bigcap_{\alpha\in I} U_\alpha = \{x \mid x \in U_\alpha \text{ for all } \alpha \in I\}$

Proof.

Example. Find the union and intersection of the collections and explain. What does the second question tell you about intersections of any collection of open sets?

$\bigcup_{n\in\mathbf{N}} [\frac{1}{n}, 1] =$ $\qquad\qquad$ $\bigcap_{n\in\mathbf{N}} (-\frac{1}{n}, \frac{1}{n}) =$

Proposition 9. Every nonempty open subset of $\mathbf{R}$ is the union of a countable collection of disjoint open intervals.

Proof.

Definition. If $E \subseteq \mathbf{R}$, we say *x is a point of closure of E* or *x is in the closure of E* if every open interval I around x contains a point in E (which may be x), that is $I \cap E \neq \emptyset$. We define the *closure of E* as

$$\overline{E} = \{x \in \mathbf{R} \mid x \text{ is a point of closure of } E\}$$

Notes. x is in the closure of E if and only if x is a cluster point of E or $x \in E$.

$E \subset \overline{E}$ and if $E \subseteq F$, then $\overline{E} \subseteq \overline{F}$.

Example. Determine the closures of the sets below.

$E = (a, b)$ $\qquad$ $F = \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\}$ $\qquad$ $G = \mathbf{Q} \cap [0, 1]$

Definition. We say the set E is *closed* if $E = \overline{E}$, that is, if E contains all its points of closure, $\overline{E} \subseteq E$.

Example. Show the sets below are closed.

$E = [a, b]$ $\qquad$ $F = \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} \cup \{0\}$ $\qquad$ $G = [0, \infty)$

Proposition 10. If $E \subseteq \mathbf{R}$, then $\overline{E}$ is closed. Furthermore, $\overline{E}$ is the smallest closed set containing E, that is, if $F \supseteq E$ and F is closed, then $\overline{E} \subseteq F$. Finally, $\overline{E} = \displaystyle\bigcap_{\substack{E \subseteq F \\ F \text{ closed}}} F$.

Proof.

Proposition 11. A subset $U \subseteq \mathbf{R}$ is open if and only if U^c is closed.

Proof.

Note. Setting $F = U^c$ the proposition says F is closed if and only if F^c is open.

Proposition 12.

1) $\emptyset$ and $\mathbf{R}$ are closed.
2) If $F_1, \ldots, F_n$ are closed, then $F_1 \cup \cdots \cup F_n$ is closed.
3) If $\{F_\alpha \mid \alpha \in I\}$ is a collection of closed sets, then $\bigcap_{\alpha \in I} F_\alpha$ is a closed set.

Proof. DeMorgan's laws and Proposition 11.

Definition. A collection of sets $\{U_\alpha, \alpha \in I\}$ is called a *cover of a set* E if $E \subseteq \bigcup_{\alpha \in I} U_\alpha$.
If additionally every U_α is open, then the cover is called an *open cover of* E. If the collection $\{U_\alpha, \alpha \in I\}$ is finite, the cover is called a *finite cover of* E. A *subcover* of a cover $\{U_\alpha, \alpha \in I\}$ is any subcollection that is still a cover of E.

Example. Are the following collections covers of $\mathbf{R}$? Are any subcovers of another one?

$\mathcal{C}_1 = \{(-\infty, n), n \in \mathbf{N}\}$ $\quad$ $\mathcal{C}_2 = \{[a, a+1), a \in \mathbf{R}\}$ $\quad$ $\mathcal{C}_3 = \{[a, a+1), a \in \mathbf{Z}\}$

$\mathcal{C}_4 = \{(a, a+1), a \in \mathbf{Z}\}$ $\quad$ $\mathcal{C}_5 = \{(a, a+1), a \in \mathbf{Q}\}$ $\quad$ $\mathcal{C}_6 = \{[2k, 2k+1), k \in \mathbf{Z}\}$

Example. Are the following collections covers of the interval $[0, 1]$ or $(0, 1)$?

$\mathcal{C}_1 = \left\{\left(-\frac{1}{n}, 1 + \frac{1}{n}\right), n \in \mathbf{N}\right\}$ $\quad$ $\mathcal{C}_2 = \left\{\left(\frac{1}{n}, 1 - \frac{1}{n}\right), n \in \mathbf{N}\right\}$ $\quad$ $\mathcal{C}_3 = \left\{\left(q, q + \frac{1}{4}\right), q \in [-1, 1] \cap \mathbf{Q}\right\}$

Definition. A subset $F \subseteq \mathbf{R}$ is called *compact* if every open cover of F has a finite subcover.

Example. The open intervals $(0, 1)$, $[0, 1)$ and $(0, 1]$ are not compact.

Example. Does this cover of $[0, 1]$ have a finite subcover: $\left\{\left(-\frac{1}{n}, 1 + \frac{1}{n}\right), n \in \mathbf{N}\right\}$?

Example. Does this cover of $[0,1]$ have a finite subcover: $\{\left(q, q+\frac{1}{4}\right), q \in [-1,2] \cap \mathbf{Q}\}$?

The Heine-Borel Theorem. A subset $F \subseteq \mathbf{R}$ is compact if and only if F is closed and bounded.

Proof.

Definition. A collection of sets $\{E_n, n \in \mathbf{N}\}$ is called

descending, if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ ascending, if $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$

The Nested Set Theorem. Let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ be a descending collection of nonempty closed sets, where F_1 is bounded. Then $\bigcap_{n \in \mathbf{N}} F_n$ is not empty.

Proof.

Definition. Let X be any set. A collection $\mathcal{A}$ of subsets of X is called a *σ-algebra* if

1) $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$.
2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.
3) If $A_n \in \mathcal{A}$ for each $n \in \mathbf{N}$, then $\bigcup_{n \in \mathbf{N}} A_n \in \mathcal{A}$.

Example. $\{X, \emptyset\}$ is a σ-algebra.

Example. $2^X = \mathcal{P}(X) = \{A \mid A \subseteq X\}$, collection of all subsets of X, is a σ-algebra.

Note. Let $\mathcal{A}$ be a σ-algebra. Then

1) If $A_n \in \mathcal{A}$, for every $n \in \mathbf{N}$, then $\bigcap_{n \in \mathbf{N}} A_n \in \mathcal{A}$.
2) If $A_1, \dots, A_n \in \mathcal{A}$, then $A_1 \cup A_2 \cup \dots \cup A_n, A_1 \cap A_2 \cap \dots \cap A_n \in \mathcal{A}$. (finite unions and intersections of elements of $\mathcal{A}$ are in $\mathcal{A}$)

Definition. Let $A_n \subseteq X$ for every $n \in \mathbf{N}$. We define

$$\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \qquad \liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

Note. The collection $\{\bigcup_{n=k}^{\infty} A_n, k \in \mathbf{N}\}$ is descending and the collection $\{\bigcap_{n=k}^{\infty} A_n, k \in \mathbf{N}\}$ is ascending. Furthermore,

$$\limsup A_n = \begin{array}{l}\text{all } x \text{ such that } x \in A_n \\ \text{for infinitely many } n\text{'s}\end{array} \qquad \liminf A_n = \begin{array}{l}\text{all } x \text{ such that } x \in A_n \\ \text{for all but finitely many } n\text{'s}\end{array}$$

If $\mathcal{A}$ is a σ-algebra and $A_n \in \mathcal{A}$ for every $n \in \mathbf{N}$, then $\limsup A_n$, $\liminf A_n \in \mathcal{A}$.

Proposition 13. Let $\mathcal{F}$ be a collection of subsets of X. Then

$$\mathcal{A} = \bigcap_{\substack{\mathcal{B} \supseteq \mathcal{F} \\ \mathcal{B} \text{ a } \sigma\text{-algebra}}} \mathcal{B} \quad \text{is a } \sigma\text{-algebra}$$

Moreover, $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{F}$, that is, if $\mathcal{A}'$ is a σ-algebra containing $\mathcal{F}$ then $\mathcal{A} \subseteq \mathcal{A}'$.

Definition. Let $\mathcal{F}$ be the collection of open subsets of $\mathbf{R}$. Then the smallest σ-algebra from Proposition 13 containing this $\mathcal{F}$ is called the *Borel sets of real numbers.* The Borel sets contain:

— G_δ-sets, countable intersections of open sets
— F_σ-sets, countable unions of closed sets
— $\limsup A_n$ and $\liminf A_n$, where each A_n is either an open or closed set.