Modern Algebra 1 — Exam 1 **MAT 513/613, Fall 2024 — D. Ivanšić**

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) For a binary operation on a set G, state what it means that the binary operation has an identity element.

Theory 2. (3pts) State the proposition that reveals what all the subgroups of $(\mathbf{Z}, +)$ are.

Theory 3. (3pts) State the fundamental theorem on cyclic groups.

Type A problems (5pts each)

A1. Write the Cayley table for $(\mathbf{Z}_6, +)$ and state the order of each element.

A2. In the group $\mathbf{Z}_{18}$ identify all 1) generators 2) elements of order 6.
Then state the generators and elements of order 6 in the cyclic group $\langle a \rangle$ where $|a| = 18$.

A3. Determine all the subgroups of $\mathbf{Z}_{12}$ and state their generators.

A4. Determine if $U(9)$ is cyclic. If it is, state all of its generators.

A5. Give a reason why $\mathbf{Q} - \{0\}$ with the operation of division is not a group.

A6. Let G be a group and $g \in G$, $g \neq e$ an element such that $|g| = |g^2|$. Show that $|g|$ is odd.

Type B problems (8pts each)

B1. Let G be the group of all bijections $\mathbf{R} \to \mathbf{R}$ (operation: composition). Show that the set of nonconstant linear functions $H = \{f : \mathbf{R} \to \mathbf{R} \mid f(x) = mx + b \text{ for some } m, b \in \mathbf{R}, m \neq 0\}$ is a subgroup of G. (Take as a given that every linear function with $m \neq 0$ is a bijection.)

B2. Let G be a group, $a, b \in G$ such that $|a| = m$, $|b| = n$ and $\gcd(m, n) = 1$. Prove that $\langle a \rangle \cap \langle b \rangle = \{e\}$. *(Hint: $\langle a \rangle \cap \langle b \rangle$ is a subgroup of $\langle a \rangle$ and of $\langle b \rangle$.)*

B3. Show that for every $n \in \mathbf{N}$, $n - 1$ is in $U(n)$ and the order of $n - 1$ is two.

B4. Let $H \leq \mathbf{Z}$ be a subgroup of $\mathbf{Z}$ with the property $3\mathbf{Z} \leq H$. Show that either $H = 3\mathbf{Z}$ or $H = \mathbf{Z}$. (In other words, there is no proper subgroup "between" $3\mathbf{Z}$ and $\mathbf{Z}$.)

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Do all the theory problems. Then do five problems, at least two of which are of type B (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the kernel of a homomorphism $\phi : G \to \overline{G}$.

Theory 2. (3pts) State the theorem about the number of 2-cycles whose composite is a given permutation. Then define an even and an odd permutation, made possible by the above theorem.

Theory 3. (3pts) Let $\phi : G \to \overline{G}$ be a homomorphism. State what we know about:
a) the relationship of $|a|$ and $|\phi(a)|$, if $a \in G$,
b) the nature of $\phi^{-1}(\overline{K})$, if $\overline{K}$ is a subgroup of $\overline{G}$.

Type A problems (5pts each)

A1. Let $\alpha \in S_6$ be given as the product of cycles: $\alpha = (162)(142)(15)$. Find the order of α.

A2. Let $\alpha \in S_n$ be a permutation whose order $|\alpha|$ is odd. Show that α is an even permutation.

A3. Let $\phi : \mathbf{Z}_8 \to \mathbf{Z}_{15}$ be given by $\phi(x) = 2x \bmod 15$. Show that ϕ is not a homomorphism.

A4. Let $\phi : \mathbf{Z}_{24} \to \mathbf{Z}_{18}$ be a homomorphism for which we know $\phi(1) = 12$.
a) Determine the image $\phi(\mathbf{Z}_{24})$. b) Determine $\ker \phi$. c) Determine $\phi^{-1}(12)$.

A5. Determine the number of automorphisms of $\mathbf{Z}_8$. Choose one that is not an identity and write its table of values.

A6. Let $\phi : \mathbf{Z} \to \mathbf{Z}$ be a homomorphism. Show that either $\phi(x) = 0$ for all x or that ϕ is injective.

Type B problems (8pts each)

B1. Let G be a group, $g \in G$ a fixed element and consider the functions $L : G \to G$ and $C : G \to G$ given by $L(x) = gx$ and $C(x) = gxg^{-1}$.
a) Show that one of L and C is a homomorphism and the other one is not.
b) Show that L and C are bijections from G onto G.

B2. Determine all the possible orders of elements of the alternating group A_5.

B3. We know that $(\mathbf{R}, +)$ is isomorphic to $(\mathbf{R}^+, \cdot)$ via the exponential function. Show that $(\mathbf{Q}, +)$ is not isomorphic to $(\mathbf{Q}^+, \cdot)$, where $\mathbf{Q}^+ = \{q \in \mathbf{Q} \mid q > 0\}$. *(Hint: if there were an isomorphism $\mathbf{Q} \to \mathbf{Q}^+$, something would get mapped to $2 \in \mathbf{Q}^+$.)*

B4. Show that S_5 contains a subgroup isomorphic to D_5 and another subgroup isomorphic to $U(7)$.

Modern Algebra 1 — Exam 3 **MAT 513/613, Fall 2024 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let $H \leq G$. Define a left coset of H in G and state the theorem on the relationship of two cosets.

Theory 2. (3pts) State Lagrange's theorem.

Theory 3. (3pts) Let G and H be groups. Define the product of groups G and H by stating the underlying set and operation.

Type A problems (5pts each)

A1. Let H and K be subgroups of a group G so that $|H| = 30$ and $|K| = 45$. What are the possibilities for $|H \cap K|$?

A2. Let $n \in \mathbf{N}$ and consider the subgroup $n\mathbf{Z}$ in $\mathbf{Z}$.
a) What are the cosets of $n\mathbf{Z}$ in $\mathbf{Z}$? How many are there?
b) Show that every element $m \in \mathbf{Z}$ is in one of the proposed cosets in a) and that all the proposed cosets are disjoint. (This proves that the collection of cosets from a) is complete.)

A3. Recall that $A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243)\}$. List the elements of the subgroup $H = \langle (243) \rangle$. Then list the elements of all the cosets of H in A_4.

A4. How many order 6 elements does $\mathbf{Z}_{12} \times \mathbf{Z}_8$ have? What are they?

A5. Write $U(90)$ as a product of cyclic groups of form $\mathbf{Z}_n$. How many elements does $U(90)$ have?

A6. Among the groups $\mathbf{Z}_{12}$, $\mathbf{Z}_2 \times \mathbf{Z}_6$ and $U(21)$, two are isomorphic. Which ones? Show that the third group is not isomorphic to the other two.

Type B problems (8pts each)

B1. Let $\mathbf{R}^* = \{x \in \mathbf{R} \mid x \neq 0\}$, $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x > 0\}$. Show that $(\mathbf{R}^*, \cdot) \approx (\mathbf{R}^+, \cdot) \times \mathbf{Z}_2$.

B2. Show that A_6 has no subgroup of order 90.

B3. Let H be the subset of the additive group $\mathbf{R} \times \mathbf{R}$, $H = \{(5t, 3t) \mid t \in \mathbf{R}\}$.
a) Show that H is a subgroup of $\mathbf{R} \times \mathbf{R}$ isomorphic to $(\mathbf{R}, +)$ by constructing an explicit isomorphism $\mathbf{R} \to H$. (Check it is a homomorphism, injective and maps onto H.)
b) Thinking of $\mathbf{R} \times \mathbf{R}$ as the plane, draw H and cosets of H.

B4. Let p and q be prime, $p < q$. Show that a group of order pq has an element of order p. *(Hint: try an example first, say a group with 65 elements.)*

B5. We know that $U(30) \approx U(5) \times U(6)$, where the isomorphism $\psi : U(5) \times U(6) \to U(30)$ can be constructed as follows:
1) Since $\gcd(5, 6) = 1$, there exist p, q such that $5p + 6q = 1$: find them.
2) Define $\psi(x, y) = (6qx + 5py) \bmod 30$. Assemble a table showing the values of $\psi(x, y)$ like a table of multiplication: in row starting with x and column starting with y, enter $\psi(x, y)$. Check ψ is a homomorphism on one example, not in general. (For a challenge, verify in general.)
3) Identify the subgroups $\psi(U(5) \times \{1\})$ and $\psi(\{1\} \times U(6))$ inside $U(30)$.