Calculus 3 — Exam 1
MAT 309, Fall 2012 — D. Ivanšić

Name: ____________________

Show all your work!

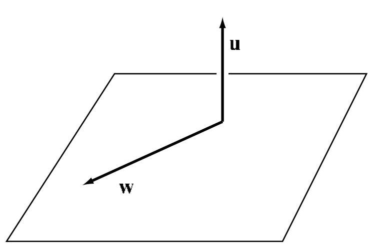
1. (18pts) Let $\mathbf{u} = \langle 3, 1, -3\rangle$ and $\mathbf{v} = \langle 0, 2, -1\rangle$.
a) Calculate $2\mathbf{u}$, $4\mathbf{u} - 3\mathbf{v}$, and $||\mathbf{u}||$.
b) Find the unit vector in direction of $\mathbf{v}$.
c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

2. (4pts) Do the coordinate systems given by the sets of vectors below (in order listed) satisfy the right hand rule?

$\{\mathbf{j}, \mathbf{k}, \mathbf{i}\}$

$\{\mathbf{i}, \mathbf{k}, \mathbf{j}\}$

3. (10pts) Vector $\mathbf{u}$ is perpendicular to the plane containing $\mathbf{w}$ (picture). Their lengths are $||\mathbf{u}|| = 3$ and $||\mathbf{w}|| = 5$. Draw a vector $\mathbf{v}$ whose angle with $\mathbf{u}$ is $\pi/3$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. What is the length of $\mathbf{v}$?



4. (12pts) Find the points of intersection of the plane $2x - 3y + 4z = 6$ with the x-, y- and z-axes and use this information to sketch the plane in a coordinate system.

5. (20pts) Two lines are given parametrically: $x = -10 - 3t$, $y = 5 + t$, $z = 10 + 2t$ and $x = -5 + 2s$, $y = -2 + 2s$, $z = 6 - s$.

a) Show that these lines intersect by finding the point of intersection.

b) Find the equation of the plane spanned by these two lines.

6. (16pts) This problem is about the surface $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

7. (10pts) Find the cylindrical coordinates of the point whose cartesian coordinates are $(-2\sqrt{3}, 2, 4)$.

8. (10pts) Sketch the following set of points given in spherical coordinates:

$0 \leq \phi \leq \dfrac{\pi}{4}, \ 1 \leq \rho \leq 3$

Bonus (10pts) Find a plane that contains the x-axis and has angle $\pi/3$ with the plane $x + y = 4$. How many such planes are there? *(Hints: recall that angle between planes is the angle between their normal vectors. Look for a unit normal vector.)*

Calculus 3 — Exam 2
MAT 309, Fall 2012 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (14pts) A curve is given by $\mathbf{r}(t) = \left\langle \cos t, \sin t, \sin \frac{t}{2} \right\rangle$, $t \in [0, 2\pi]$.
a) Sketch this curve.
b) Find the parametric equation of the tangent line to the curve at time $t = \frac{\pi}{2}$ and draw this tangent line on your sketch.

2. (16pts) Consider the curve C that is the intersection of the cylinder $x^2 + z^2 = 4$ with the parabolic cylinder $z = y^2$.
a) Sketch a picture.
b) Parametrize each of the two parts of the curve corresponding to $x \geq 0$ and $x \leq 0$, taking y as the parameter.
c) What interval does the parameter run through to get each of the two parts?

3. (22pts) Consider the function $f(x, y) = \dfrac{y}{x^2}$ for $x > 0$, y any.

a) Identify and draw vertical traces for this function.

b) Using a), draw the graph of the function.

c) Draw a rough contour map for the function, with contour interval $\frac{1}{2}$, going from $c = -\frac{3}{2}$ to $c = \frac{3}{2}$.

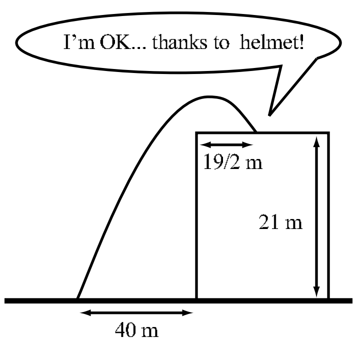
d) By looking at the contour map, indicate a path on which we could move from $(\sqrt{2}, 1)$ in order to increase the value of the function to 1.

4. (16pts) Find the length of the curve with the parametrization $\mathbf{r}(t) = \left\langle t^{\frac{3}{2}}, 5\sin t, 5\cos t \right\rangle$, $t \in [0, 4\pi]$.

5. (20pts) Acting on a dare, your favorite physics professor Dr. __________ (insert name here) launches himself from 40 meters away from base of Faculty Hall (height 21 meters) and lands on its roof $\frac{19}{2}$ meters away from the edge. (See picture.) The angle α of launch was such that $\cos\alpha = \frac{3}{5}$. Assume $g = 10$ for simplicity.

a) Find his position at time t. The expression will have an unknown initial speed v_0 in it.

b) Now find v_0.



6. (12pts) Determine and sketch the domain of the function $f(x, y) = \dfrac{\sqrt{x - y - 5}}{x + y}$.

Bonus (10pts) Use coordinates to prove the formula $(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ for any two vector functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$,

Calculus 3 — Exam 3
MAT 309, Fall 2012 — D. Ivanšić

Name:____________________________

Show all your work!

1. (12pts) Find the equation of the tangent plane to the surface $x^2 - \frac{y^2}{4} - \frac{z^2}{9} = 1$ at the point $\left(2, \sqrt{2}, 3\sqrt{\frac{5}{2}}\right)$. Simplify the equation to standard form.

2. (18pts) Let $f(x, y) = x^2 - y^2$.
a) Find the directional derivative of f at the point $(3, 2)$ in the direction of $\mathbf{v} = \langle 1, -5 \rangle$.
b) At the point $(3, 2)$, in which direction does the function increase the fastest? Decrease the fastest?
c) Find $\frac{d}{dt} f(\mathbf{r}(t))$ for the path $\mathbf{r}(t) = \langle 4 - 3t, t^2 - 2t \rangle$ when $t = 1$.

3. (20pts) Let $f(x, y) = \dfrac{\ln x}{\ln x + \ln y}$, $x = e^u \cos v$, $y = e^u \sin v$. Use the chain rule to find $\dfrac{\partial f}{\partial v}$ when $u = 3$, $v = \dfrac{\pi}{6}$.

4. (14pts) In an improbable scenario, you find yourself on a desert island with no technology and have to estimate $\dfrac{\sqrt{20.2}}{\sqrt{4.99}}$. (Or, in a more likely one, you find yourself in test-taking environment with no calculators allowed.) Use linearization to estimate this number, and compare it to the calculator result of 2.011988.

5. (14pts) Using implicit differentiation, find $\dfrac{\partial z}{\partial y}$ at the point $(-1, 2, 1)$, if $x^2y+y^2z+z^2x = 5$.

6. (22pts) Find and classify the local extremes for $f(x, y) = \frac{2}{3}x^3 + 2xy - 2y^2 - 10x$.

Bonus (10pts) Find the absolute maximum and minimum of $f(x, y) = x^2 + y^2 - 6x - 4y$ on the domain $x \geq 0$, $y \geq 0$, $y \leq -2x + 3$.

Calculus 3 — Exam 4
MAT 309, Fall 2012 — D. Ivanšić

Name: ____________________

Show all your work!

1. (18pts) Find $\iint_D x + y\, dA$ if D is the triangle with vertices $(0,0)$, $(1,1)$, $(2,1)$. Sketch the region of integration.

2. (18pts) Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} (4-y^2)^{\frac{3}{2}}\, dy\, dx$ by changing the order of integration. Sketch the region of integration.

3. (16pts) Use polar coordinates to find the area of the region that is inside the unit circle and to the right of the line $x = \frac{1}{2}$. Sketch the region of integration first.

4. (16pts) Sketch the region W that is in the first octant $(x, y, z \geq 0)$, above the plane $y = z$ and below the plane $x + y + z = 1$. Then write the two iterated triple integrals that stand for $\iiint_W f\, dV$ which end in $dx\, dz\, dy$, and $dz\, dy\, dx$.

5. (16pts) Use cylindrical coordinates to set up $\iiint_W \frac{x+y+z}{x^2+y^2}\,dV$ where W is the part of the region above the paraboloid $z = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 12$ where $x \geq 0$. Sketch the region of integration. Do not evaluate the integral.

6. (16pts) Use change of variables to find the integral $\iint_D (2x+y)^2\,dA$ if D is the rhombus bounded by $y = 2x + 2$, $y = 2x - 2$, $y = -2x + 2$, $y = -2x - 2$. Sketch the region D.

Bonus. (10pts) The intersection of balls $x^2 + y^2 + z^2 \leq 1$ and $x^2 + y^2 + (z-1)^2 \leq 1$ is a lens-shaped region. Find its volume by doing the following:
a) Use spherical coordinates to find the volume of the region inside $x^2 + y^2 + (z-1)^2 \leq 1$ that is outside of $x^2 + y^2 + z^2 \leq 1$.
b) Find the volume of the lens using your result from a). Recall that the volume of a ball of radius R is $\frac{4}{3}\pi R^3$.

Calculus 3 — Exam 5
MAT 309, Fall 2012 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (15pts) Let $\mathbf{F}(x, y) = \langle y, x \rangle$.
a) Roughly draw the vector field $\mathbf{F}(x, y)$, scaling the vectors for a better picture.
b) Guess a function $\phi(x, y)$ so that $\mathbf{F} = \nabla\phi$.
c) How could you have roughly done a) without evaluating the vector field at various points?
d) What is $\int_C \mathbf{F} \cdot d\mathbf{s}$ if C is part of the curve $y = \sin x$ from $(0, 0)$ to $(\frac{\pi}{2}, 1)$? How about if C is a straight line segment from $(0, 0)$ to $(\frac{\pi}{2}, 1)$?
e) What is $\int_C \mathbf{F} \cdot d\mathbf{s}$ if C is the unit circle?

2. (15pts) Let C be the curve $x = 1 + t$, $y = 4 \sin t$, $z = t^2$, for $t \in [0, \pi]$.
a) Set up $\int_C z(e^x + e^y)\, ds$.
b) Set up $\int_C \mathbf{F} \cdot d\mathbf{s}$, if $\mathbf{F}(x, y, z) = \langle x^2, z, y^2 \rangle$.
In both cases simplify the set-up, but do not evaluate the integral.

3. (16pts) One of the two vectors fields below is not a gradient field, and the other one is (cross partials, remember?). Identify which is which, and find the potential function for the one that is.

$\mathbf{F}(x, y, z) = \langle \cos(xz), \sin(yz), xy \sin z \rangle$ $\mathbf{G}(x, y, z) = \langle 2xy + z^2, x^2 + 2yz, y^2 + 2xz \rangle$

4. (12pts) A surface is parametrized by $\Phi(u, v) = (u^2 + v^2, uv, u^2 - v^2)$. Find the equation of the tangent plane to this surface at the point where $(u, v) = (1, 1)$.

5. (24pts) Find the surface integral $\iint_S x\, dS$, if S is part of the sphere $x^2 + y^2 + z^2 = 4$ that is in the octant $x, y, z \geq 0$. Draw the surface, parametrize it and specify the planar region D where your parameters come from.

6. (18pts) Set up the integral for $\iint_S \mathbf{F} \cdot d\mathbf{S}$, if S is the part of the paraboloid $z = 10 - x^2 - y^2$ that is above the xy-plane and $\mathbf{F}(x, y, z) = \langle x, y, 1 + z \rangle$. (The surface does not include any part of the xy-plane, just part of the paraboloid.) Use the normal vectors to the paraboloid that point upwards. Draw the surface and some normal vectors, parametrize the surface and specify the planar region D where your parameters come from. Simplify the set-up, but do not evaluate the integral.

Bonus. (10pts) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 9$ that is between the planes $z = 1$ and $z = 2$.

Calculus 3 — Final Exam **MAT 309, Fall 2012 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

1. (6pts) What is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ if we know that $||\mathbf{a}|| = 3$, $||\mathbf{b}|| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 6$?

2. (10pts) Find the equation of the plane that contains the points $A = (1, 3, 1)$, $B = (4, -7, 2)$ and $C = (-1, -1, -1)$.

3. (16pts) Consider the function $f(x, y) = \dfrac{y}{x}$ for $x, y > 0$.
c) Draw a rough contour map for the function, with contour interval $\frac{1}{4}$, going from $c = \frac{1}{4}$ to $c = \frac{3}{2}$.
b) Find ∇f and roughly draw this vector field (scale the vectors for a better picture) Note that no computation is needed to draw the vector field.
c) What is $\int_C \nabla f \cdot d\mathbf{s}$ if C is the arc of the parabola $y = x^2$ from $(1, 1)$ to $(3, 9)$?

4. (14pts) A curve is given by $\mathbf{r}(t) = \langle \cos t, \sin t, 5t \rangle$, $t \in [0, 4\pi]$.
a) Sketch this curve.
b) Find the parametric equation of the tangent line to the curve at time $t = \pi$ and draw this tangent line on your sketch.

5. (10pts) Find the equation of the tangent plane to the surface $\frac{x^2}{12} - \frac{y^2}{4} + \frac{z^2}{2} = 1$ at the point $(3, 1, -1)$. Simplify the equation to standard form.

6. (14pts) Let $f(x, y) = \dfrac{e^{xy}}{x^2}$, $x = -2u + 5v$, $y = u - 3v$. Use the chain rule to find $\dfrac{\partial f}{\partial u}$ when $u = 6$, $v = 2$.

7. (16pts) Find and classify the local extremes for $f(x, y) = 2x^2y^2 + 3x^3 - 4x$.

8. (18pts) Find $\iint_D x^2 + y^2 \, dA$ if D is the region above $y = |x|$ and between lines $y = 3$ and $y = 5$. Sketch the region of integration.

9. (12pts) Sketch the region W that is the part of the ball $x^2 + y^2 + z^2 \leq 16$, above the plane $z = 3$, and to the right of the plane $y = 2$. Then write the iterated triple integral that stands for $\iiint_W f \, dV$ which ends in $dy \, dz \, dx$.

10. (22pts) Compute the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, if S is the part of the paraboloid $z = 10 - x^2 - y^2$ that is above the xy-plane and $\mathbf{F}(x, y, z) = \langle x, y, 1 + z \rangle$. (The surface does not include any part of the xy-plane, just part of the paraboloid.) Use the normal vectors to the paraboloid that point upwards. Draw the surface and some normal vectors, parametrize the surface and specify the planar region D where your parameters come from.

11. (12pts) Is the vector field below a gradient field? If yes, find its potential function.

$\mathbf{F}(x, y, z) = \langle -z^3, 3z, 2z + 3y - 3xz^2 \rangle$

Bonus. (15pts) The intersection of balls $x^2 + y^2 + z^2 \leq 1$ and $x^2 + y^2 + (z-1)^2 \leq 1$ is a lens-shaped region. Find its volume by doing the following:
a) Use spherical coordinates to find the volume of the region inside $x^2 + y^2 + (z-1)^2 \leq 1$ that is outside of $x^2 + y^2 + z^2 \leq 1$.
b) Find the volume of the lens using your result from a). Recall that the volume of a ball of radius R is $\frac{4}{3}\pi R^3$.