Calculus 3 — Final Exam
MAT 309, Fall 2012 — D. Ivanšić

Name: Saul Ocan

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1. (6pts) What is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ if we know that $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 6$?

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ so $\cos\theta = \frac{1}{2}$

$6 = 3 \cdot 4 \cos\theta$ $\theta = \frac{\pi}{3}$

2. (10pts) Find the equation of the plane that contains the points $A = (1, 3, 1)$, $B = (4, -7, 2)$ and $C = (-1, -1, -1)$.

$\vec{r} \cdot \vec{n} = \vec{r_0} \cdot \vec{n}$

$\vec{AB} = \langle 3, -10, 1 \rangle$

$\vec{AC} = \langle -2, -4, -2 \rangle$, use $\langle 1, 2, 1 \rangle$

$6x + y - 8z = 6 + 3 - 8$

$6x + y - 8z = 1$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -10 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -12\vec{i} - 2\vec{j} + 16\vec{k}$$

use $\langle 6, 1, -8 \rangle = \vec{n}$

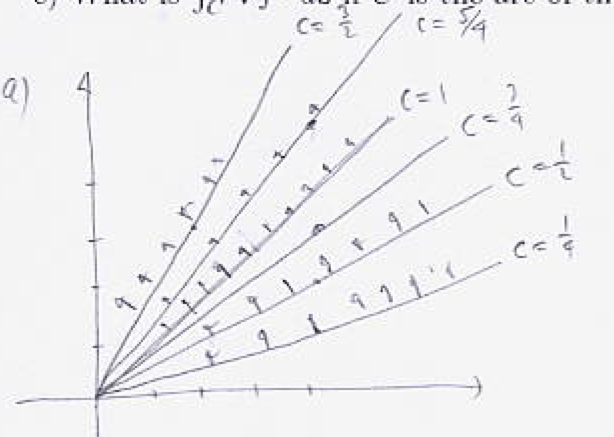
3. (16pts) Consider the function $f(x, y) = \dfrac{y}{x}$ for $x, y > 0$.

a) Draw a rough contour map for the function, with contour interval $\frac{1}{4}$, going from $c = \frac{1}{4}$ to $c = \frac{3}{2}$.

b) Find ∇f and roughly draw this vector field (scale the vectors for a better picture) Note that no computation is needed to draw the vector field.

c) What is $\int_C \nabla f \cdot d\mathbf{s}$ if C is the arc of the parabola $y = x^2$ from $(1, 1)$ to $(3, 9)$?

a)



$\frac{y}{x} = c$

$y = cx$

b) $\nabla f = \langle -\frac{y}{x^2}, \frac{1}{x} \rangle$ ∇f are perpendicular to level curves

c) $\int_C \nabla f \cdot d\vec{s} = f(3,9) - f(1,1)$

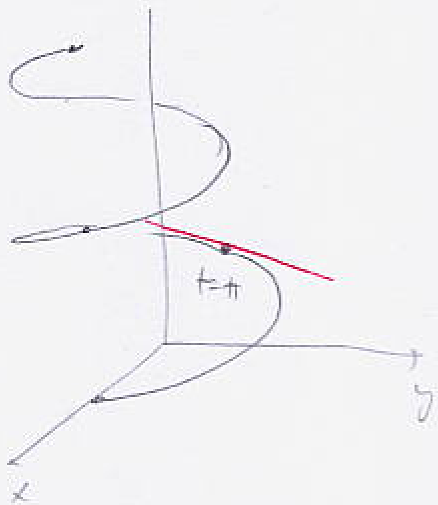
$= \frac{9}{3} - \frac{1}{1} = 3 - 1 = 2$

4. (14pts) A curve is given by $\mathbf{r}(t) = \langle \cos t, \sin t, 5t \rangle$, $t \in [0, 4\pi]$.
a) Sketch this curve.
b) Find the parametric equation of the tangent line to the curve at time $t = \pi$ and draw this tangent line on your sketch.

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 5 \rangle$

$\vec{r}\,'(\pi) = \langle 0, -1, 5 \rangle$

$\vec{r}(\pi) = \langle -1, 0, 5\pi \rangle$



It's a helix along z-axis

Equation of tangent line:

$x = -1$

$y = -t$

$z = 5\pi + 5t$

5. (10pts) Find the equation of the tangent plane to the surface $\frac{x^2}{12} - \frac{y^2}{4} + \frac{z^2}{2} = 1$ at the point $(3, 1, -1)$. Simplify the equation to standard form.

$f(x,y,z) = \frac{x^2}{12} - \frac{y^2}{4} + \frac{z^2}{2} = 1$

$\nabla f = \left\langle \frac{2x}{12}, -\frac{2y}{4}, \frac{2z}{2} \right\rangle$

$= \left\langle \frac{x}{6}, -\frac{y}{2}, z \right\rangle$

$\nabla f(3,1,-1) = \left\langle \frac{1}{2}, -\frac{1}{2}, -1 \right\rangle$

Take $\vec{n} = \langle 1, -1, -2 \rangle = 2\nabla f$

$\vec{r} \cdot \vec{n} = \vec{r}_0 \cdot \vec{n}$

$x - y - 2z = 3 - 1 + 2$

$x - y - 2z = 4$

6. (14pts) Let $f(x,y) = \frac{e^{xy}}{x^2}$, $x = -2u + 5v$, $y = u - 3v$. Use the chain rule to find $\frac{\partial f}{\partial u}$ when $u = 6$, $v = 2$.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{e^{xy}\cdot y\cdot x^2 - e^{xy}\cdot 2x}{x^4}(-2) + \frac{e^{xy}\cdot x}{x^2}\,1$$

$$= -2\frac{e^{xy}(xy-2)}{x^3} + \frac{e^{xy}}{x}$$

When $u=6$, $x=-2$
$v=2$, $y=0$

$$\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(6,2)} = -2\frac{e^0(0-2)}{(-2)^3} + \frac{e^0}{(-2)} = \frac{4}{-8} + \frac{1}{-2} = -1$$

7. (16pts) Find and classify the local extremes for $f(x,y) = 2x^2y^2 + 3x^3 - 4x$.

$$\frac{\partial f}{\partial x} = 4xy^2 + 9x^2 - 4$$

$$\frac{\partial f}{\partial y} = 4x^2y$$

$$\begin{cases} 4xy^2 + 9x^2 - 4 = 0 \\ 4x^2y = 0 \end{cases}$$

2nd eq. says: $x = 0$ or $y = 0$

Gives in first eq. $-4=0$ no sol.

$9x^2 - 4 = 0$

$x^2 = \frac{4}{9}$

$x = \pm\frac{2}{3}$

Candidates: $\left(\frac{2}{3}, 0\right), \left(-\frac{2}{3}, 0\right)$

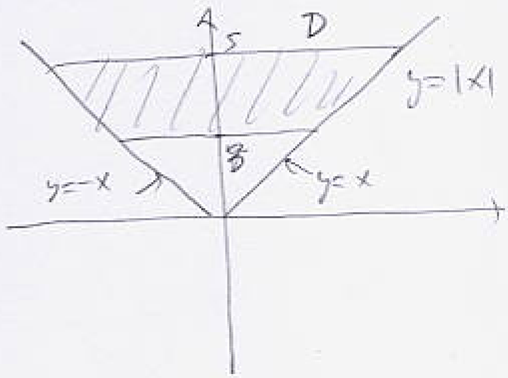
$$D = \begin{vmatrix} 4y^2 + 18x & 8xy \\ 8xy & 4x^2 \end{vmatrix}$$

$$D\left(\tfrac{2}{3}, 0\right) = \begin{vmatrix} 12 & 0 \\ 0 & 4\cdot\frac{4}{9} \end{vmatrix} > 0$$

Since $f_{xx} > 0$, have local min at $\left(\frac{2}{3}, 0\right)$

$$D\left(-\tfrac{2}{3}, 0\right) = \begin{vmatrix} -12 & 0 \\ 0 & 4\cdot\frac{4}{9} \end{vmatrix} < 0$$ so $\left(-\frac{2}{3}, 0\right)$ is a saddle point.

8. (18pts) Find $\iint_D x^2 + y^2\, dA$ if D is the region above $y = |x|$ and between lines $y = 3$ and $y = 5$. Sketch the region of integration.



Views as horizontally simple

$$\iint_D x^2+y^2\, dA = \int_3^5 \int_{-y}^{y} x^2+y^2\, dx\, dy$$

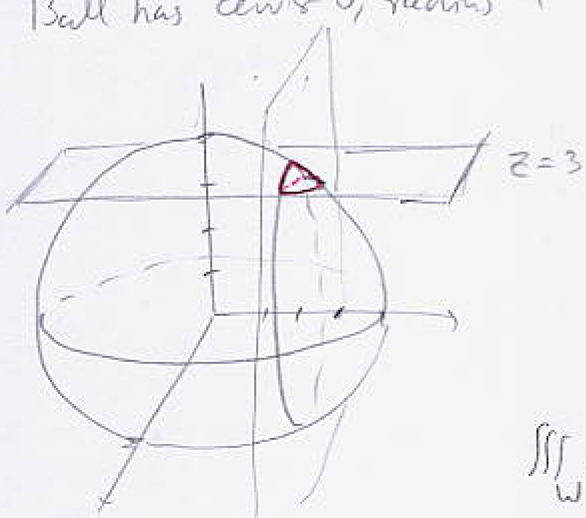
$$= \int_3^5 \frac{x^3}{3}\Big|_{-y}^{y} + y^2(y-(-y))\, dy = \int_3^5 \frac{1}{3}\left(y^3-(-y)^3\right) + 2y^3\, dy$$

$$= \int_3^5 \frac{2}{3}y^3 + 2y^3\, dy = \frac{8}{3}\int_3^5 y^3\, dy = \frac{8}{3}\,\frac{y^4}{4}\Big|_3^5$$

$$= \frac{2}{3}\left(5^4-3^4\right) = \frac{2}{3}(625-81) = \frac{2}{3}\cdot 544 = \frac{1088}{3}$$

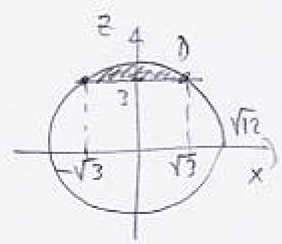
9. (12pts) Sketch the region W that is the part of the ball $x^2 + y^2 + z^2 \le 16$, above the plane $z = 3$, and to the right of the plane $y = 2$. Then write the iterated triple integral that stands for $\iiint_W f\, dV$ which ends in $dy\, dz\, dx$.

Ball has center 0, radius 4



Proj. to xz plane:

$y=2 \Rightarrow x^2+z^2=12$



When $z=3$

$x^2+9=12$

$x^2=3$

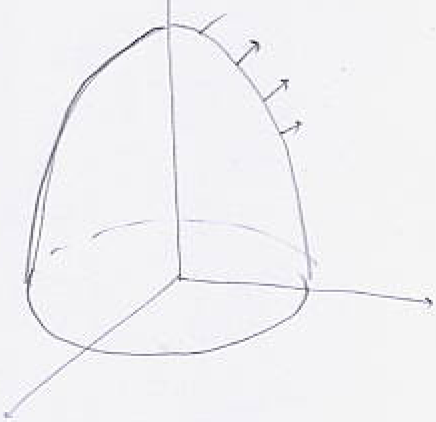
$x=\pm\sqrt{3}$

$$\iiint_W f\, dV = \iint_D \int_2^{\sqrt{16-x^2-z^2}} f\, dA$$

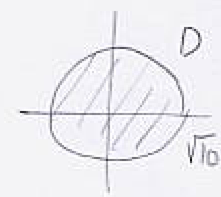
$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_3^{\sqrt{12-y^2}} \int_2^{\sqrt{16-x^2-z^2}} f\ dy\, dz\, dx$$

10. (22pts) Compute the integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, if S is the part of the paraboloid $z = 10 - x^2 - y^2$ that is above the xy-plane and $\mathbf{F}(x, y, z) = \langle x, y, 1+z\rangle$. (The surface does not include any part of the xy-plane, just part of the paraboloid.) Use the normal vectors to the paraboloid that point upwards. Draw the surface and some normal vectors, parametrize the surface and specify the planar region D where your parameters come from.

$z = 10 - x^2 - y^2$



Param. $x = u$
$y = v$
$z = 10 - u^2 - v^2$



Proj. to xy-plane: $10 - x^2 - y^2 = 0$
$x^2 + y^2 = 10$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1\rangle$$

correct normal since z-coord is positive

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{T}_u \times \vec{T}_v)\, dA$$

$$= \iint_D \langle u, v, 1+10-u^2-v^2\rangle \cdot \langle 2u, 2v, 1\rangle\, dA$$

$$= \iint_D 2u^2 + 2v^2 + 11 - u^2 - v^2\, dA = \iint_D u^2 + v^2 + 11\, dA$$

$$= \left[\begin{array}{c}\text{change to} \\ \text{polar coord.} \\ \text{in } uv\text{-plane}\end{array}\right] = \int_0^{2\pi}\int_0^{\sqrt{10}} (r^2+11)\, r\, dr\, d\theta = 2\pi\left(\frac{r^4}{4} + \frac{11r^2}{2}\right)\Big|_0^{\sqrt{10}}$$

$$= 2\pi\left(\frac{100}{4} + \frac{11\cdot 10}{2}\right) = 2\pi(25 + 55) = 160\pi$$

11. (12pts) Is the vector field below a gradient field? If yes, find its potential function.

$\mathbf{F}(x, y, z) = \langle -z^3, 3z, 2z + 3y - 3xz^2 \rangle$

$$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}$$

$$\frac{\partial F_1}{\partial z} = -3z^2 = \frac{\partial F_3}{\partial x}$$

$$\frac{\partial F_2}{\partial z} = 3 = \frac{\partial F_3}{\partial y}$$

Since cross-partials are same, locally there is a potential function φ

$$\frac{\partial \varphi}{\partial x} = -z^3 \quad \text{so} \quad \varphi = -z^3 x + g(y,z)$$

$$3z = \frac{\partial \varphi}{\partial y} = 0 + \frac{\partial g}{\partial y} \quad \text{so} \quad \frac{\partial g}{\partial y} = 3z$$

$$g = 3yz + h(z), \quad \varphi = -z^3x + 3yz + h(z)$$

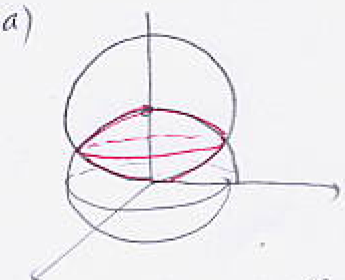
$$2z + 3y - 3xz^2 = \frac{\partial \varphi}{\partial z} = -3z^2x + 3y + h'(z)$$

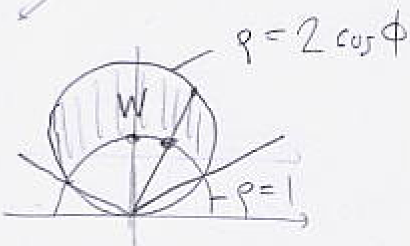
$$\text{so} \quad h'(z) = 2z, \quad h(z) = z^2 + C$$

$$\boxed{\varphi = -z^3x + 3yz + z^2 + C}$$

Bonus. (15pts) The intersection of balls $x^2 + y^2 + z^2 \leq 1$ and $x^2 + y^2 + (z-1)^2 \leq 1$ is a lens-shaped region. Find its volume by doing the following:
a) Use spherical coordinates to find the volume of the region inside $x^2 + y^2 + (z-1)^2 \leq 1$ that is outside of $x^2 + y^2 + z^2 \leq 1$.
b) Find the volume of the lens using your result from a). Recall that the volume of a ball of radius R is $\frac{4}{3}\pi R^3$.

a)





$$2\cos\phi = 1, \quad \cos\phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}, -\frac{\pi}{3} \text{ not in } [0, \pi]$$

$$V = \iiint_W 1\, dV = \int_0^{2\pi} \underbrace{\int_0^{\frac{\pi}{3}} \int_1^{2\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi}_{\text{const. for } \theta} \, d\theta$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{3}} \sin\phi \left. \frac{\rho^3}{3} \right|_1^{2\cos\phi} = \frac{2\pi}{3} \int_0^{\pi/3} \sin\phi \left(8\cos^3\phi - 1\right) d\phi$$

$$= \left[\begin{array}{ll} u = \cos\phi & \phi = \frac{\pi}{3},\ u = \frac{1}{2} \\ du = -\sin\phi\, d\phi & \phi = 0,\ u = 1 \\ -du = \sin\phi\, d\phi & \end{array} \right] = \frac{2\pi}{3} \int_1^{\frac{1}{2}} (8u^3 - 1)(-du)$$

$$= \frac{2\pi}{3} \int_{\frac{1}{2}}^1 8u^3 - 1 \, du = \frac{2\pi}{3} \left(\left. \frac{8u^4}{4} \right|_{\frac{1}{2}}^1 - \frac{1}{2} \right) =$$

$$= \frac{2\pi}{3} \left(2\left(1 - \frac{1}{16}\right) - \frac{1}{2} \right) = \frac{2\pi}{3} \left(\frac{15}{8} - \frac{1}{2} \right) = \frac{2\pi}{3} \cdot \frac{11}{8} = \boxed{\frac{11\pi}{12}}$$

b) Vol. of lens $= \frac{4}{3}\pi \cdot 1^3 - \frac{11\pi}{12} = \pi\left(\frac{4}{3} - \frac{11}{12}\right) = \boxed{\frac{5\pi}{12}}$