Calculus 3 — Exam 5
MAT 309, Fall 2012 — D. Ivanšić

Name: Saul Ocean

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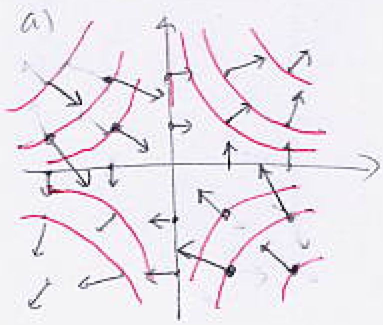
1. (15pts) Let $\mathbf{F}(x,y) = \langle y, x\rangle$.
a) Roughly draw the vector field $\mathbf{F}(x,y)$, scaling the vectors for a better picture.
b) Guess a function $\phi(x,y)$ so that $\mathbf{F} = \nabla\phi$.
c) How could you have roughly done a) without evaluating the vector field at various points?
d) What is $\int_C \mathbf{F}\cdot d\mathbf{s}$ if C is part of the curve $y = \sin x$ from $(0,0)$ to $(\frac{\pi}{2}, 1)$? How about if C is a straight line segment from $(0,0)$ to $(\frac{\pi}{2}, 1)$?
e) What is $\int_C \mathbf{F}\cdot d\mathbf{s}$ if C is the unit circle?

a)



b) $\phi(x,y) = xy$

c) $\nabla\varphi$ is perpendicular to level curves of $\varphi(x,y) = xy$

d) $\int_C \vec{F}\cdot d\vec{r} = \varphi(\frac{\pi}{2},1) - \varphi(0,0) = \frac{\pi}{2}$

Same answer for line segment

e) $\int_C \vec{F}\cdot d\vec{s} = \varphi(P) - \varphi(P) = 0$

2. (15pts) Let C be the curve $x = 1+t$, $y = 4\sin t$, $z = t^2$, for $t \in [0,\pi]$.
a) Set up $\int_C z(e^x + e^y)\,ds$.
b) Set up $\int_C \mathbf{F}\cdot d\mathbf{s}$, if $\mathbf{F}(x,y,z) = \langle x^2, z, y^2\rangle$.
In both cases simplify the set-up, but do not evaluate the integral.

$x' = 1$
$y' = 4\cos t$
$z' = 2t$

$\sqrt{(x')^2+(y')^2+(z')^2} =$

$= \sqrt{1+16\cos^2 t + 4t^2}$

a) $\int_C z(e^x+e^y)\,ds = \int_0^{\pi} t^2(e^{1+t} + e^{4\sin t})\sqrt{1+4t^2+16\cos^2 t}\,dt$

b) $\int \vec{F}\cdot d\vec{r} = \int_0^{\pi} \langle (1+t)^2, t^2, 16\sin^2 t\rangle \cdot \langle 1, 4\cos t, 2t\rangle\,dt$

$= \int_0^{\pi} (1+t)^2 + 4t^2\cos t + 32t\sin^2 t\,dt$

3. (16pts) One of the two vectors fields below is not a gradient field, and the other one is (cross partials, remember?). Identify which is which, and find the potential function for the one that is.

$\mathbf{F}(x, y, z) = \langle \cos(xz), \sin(yz), xy\sin z \rangle$

$\mathbf{G}(x, y, z) = \langle 2xy + z^2, x^2 + 2yz, y^2 + 2xz \rangle$

$$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial F_1}{\partial z} &= -\sin(xz)\cdot x \\ \frac{\partial F_3}{\partial x} &= y\sin z \end{aligned}\right\}$$ not equal, so $\vec{F}$ is not a gradient field

$$\left.\begin{aligned} \frac{\partial G_1}{\partial y} &= 2x \quad \frac{\partial G_2}{\partial x} = 2x \\ \frac{\partial G_1}{\partial z} &= 2z \quad \frac{\partial G_3}{\partial x} = 2z \\ \frac{\partial G_2}{\partial z} &= 2y \quad \frac{\partial G_3}{\partial y} = 2y \end{aligned}\right\}$$ Cross-partials are equal, so there is a potential function

$$\frac{\partial \varphi}{\partial x} = 2xy + z^2 \text{ so } \varphi = x^2y + z^2x + g(y,z)$$

$$x^2 + 2yz = \frac{\partial \varphi}{\partial y} = x^2 + \frac{\partial g}{\partial y}(y,z)$$

so $\frac{\partial g}{\partial y} = 2yz$, $g = y^2z + h(z)$

$$y^2 + 2xz = \frac{\partial \varphi}{\partial z} = 2zx + y^2 + h'(z)$$

We get $h'(z) = 0$, so $h(z) = C$

$$\varphi(x,y,z) = x^2y + z^2x + y^2z + C$$

4. (12pts) A surface is parametrized by $\Phi(u, v) = (u^2 + v^2, uv, u^2 - v^2)$. Find the equation of the tangent plane to this surface at the point where $(u, v) = (1, 1)$.

$$\frac{\partial \Phi}{\partial u} = \langle 2u, v, 2u \rangle \qquad \langle 2,1,2 \rangle$$

eval. at (1,1) give

$$\frac{\partial \Phi}{\partial v} = \langle 2v, u, -2v \rangle \qquad \langle 2,1,-2 \rangle$$

$\Phi(1,1) = (2,1,0)$

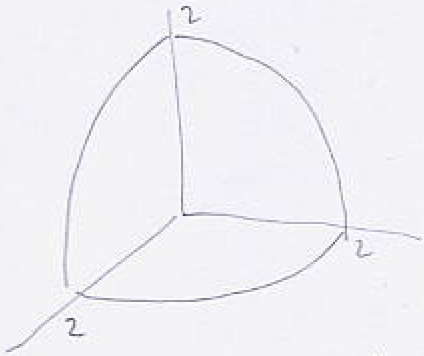
$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = -4\vec{i} - (-8\vec{j}) + 0 = -4\vec{i} + 8\vec{j},$$ may use $\vec{i} - 2\vec{j}$ as normal

$$\vec{r}\cdot\vec{n} = \vec{r}_0\cdot\vec{n}$$

Tangent plane: $x - 2y = 2 - 2$

$$\boxed{x - 2y = 0}$$

5. (24pts) Find the surface integral $\iint_S x\,dS$, if S is part of the sphere $x^2+y^2+z^2=4$ that is in the octant $x, y, z \geq 0$. Draw the surface, parametrize it and specify the planar region D where your parameters come from.



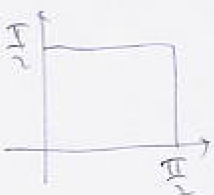
Parametrize using polar coordinates.

$$\rho = 2 \qquad x = 2\sin\phi\cos\theta$$
$$0 \leq \phi \leq \tfrac{\pi}{2} \qquad y = 2\sin\phi\sin\theta$$
$$0 \leq \theta \leq \tfrac{\pi}{2} \qquad z = 2\cos\phi$$

$$\vec{T}_\theta \times \vec{T}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \end{vmatrix}$$

$$= \left\langle -4\sin^2\phi\cos\theta,\ -4\sin^2\phi\sin\theta,\ -4\sin\phi\cos\phi\sin^2\theta - 4\sin\phi\cos\phi\cos^2\theta \right\rangle$$

D =



$$= -4\sin\phi\left\langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \right\rangle$$

$$\|\vec{T}_\theta \times \vec{T}_\phi\| = |4\sin\phi|\sqrt{\underbrace{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}_{\sin^2\phi} + \cos^2\phi} = |4\sin\phi|$$

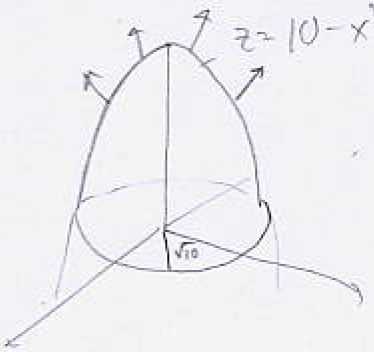
$$\iint_S x\,dS = \iint_D 2\sin\phi\cos\theta \cdot \underbrace{|4\sin\phi|}_{>0 \text{ on } D}\,dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 8\sin^2\phi\cos\theta\,d\phi\,d\theta = 8\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta \cdot \int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi$$

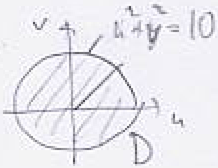
$$= 8\cdot\left(\sin\theta\Big|_0^{\pi/2}\right)\cdot\int_0^{\pi/2}\frac{1-\cos 2\phi}{2}\,d\phi$$

$$= 8\cdot 1\cdot \frac{1}{2}\left(\frac{\pi}{2} - \underbrace{\frac{\sin 2\theta}{2}\Big|_0^{\pi/2}}_{\frac{\sin\pi}{2} - \frac{\sin 0}{2} = 0}\right) = 2\pi$$

6. (18pts) Set up the integral for $\iint_S \mathbf{F}\cdot d\mathbf{S}$, if S is the part of the paraboloid $z = 10 - x^2 - y^2$ that is above the xy-plane and $\mathbf{F}(x, y, z) = \langle x, y, 1 + z\rangle$. (The surface does not include any part of the xy-plane, just part of the paraboloid.) Use the normal vectors to the paraboloid that point upwards. Draw the surface and some normal vectors, parametrize the surface and specify the planar region D where your parameters come from. Simplify the set-up, but do not evaluate the integral.



$x = u$
$y = v$
$z = 10 - u^2 - v^2$



$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \underbrace{\langle 2u, 2v, 1\rangle}$$

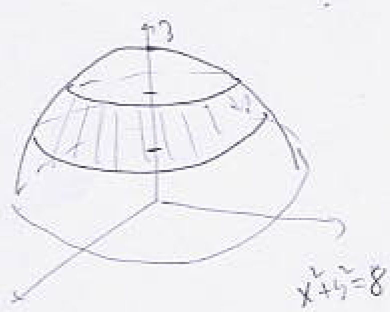
point in correct direction, (upwards), since z is positive

$$\iint \vec{F}\cdot d\vec{S} = \iint_D \langle u, v, 1+10-u^2-v^2\rangle \cdot \langle 2u, 2v, 1\rangle \, dA$$

$$= \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-\sqrt{10-u^2}}^{\sqrt{10-u^2}} 2u^2 + 2v^2 + 11 - u^2 - v^2 \, dv\, du$$

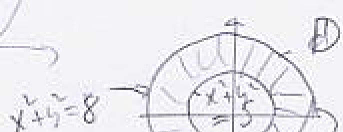
$$= \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-\sqrt{10-u^2}}^{\sqrt{10-u^2}} (u^2 + v^2 + 11) \, dv\, du$$

Bonus. (10pts) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 9$ that is between the planes $z = 1$ and $z = 2$.



$x = u$
$y = v$
$z = \sqrt{9 - x^2 - y^2}$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{x}{\sqrt{9-u^2-v^2}} \\ 0 & 1 & -\frac{y}{\sqrt{9-x^2-y^2}} \end{vmatrix} = \left\langle \frac{x}{\sqrt{9-u^2-v^2}}, \frac{y}{\sqrt{9-u^2-v^2}}, 1 \right\rangle$$



$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{\frac{u^2}{9-u^2-v^2} + \frac{v^2}{9-u^2-v^2} + 1} = \sqrt{\frac{9}{9-u^2-v^2}} = \frac{3}{\sqrt{9-u^2-v^2}}$$

$x^2+y^2+2^2=9$
$x^2+y^2=5$

$x^2+y^2+1^2=9$
$x^2+y^2=8$

$$\iint_S 1\, dS = \iint_D \frac{3}{\sqrt{9-u^2-v^2}}\, dA = \begin{bmatrix} \text{convert to} \\ \text{polar in} \\ uv\text{-plane} \end{bmatrix} = \int_0^{2\pi} \int_{\sqrt{5}}^{\sqrt{8}} \frac{3}{\sqrt{9-r^2}}\, r\, dr\, d\theta$$

$$= 2\pi\left(-3\sqrt{9-r^2}\right)\Big|_{\sqrt{5}}^{\sqrt{8}} = -6\pi\left(\sqrt{1} - \sqrt{4}\right) = 6\pi.$$