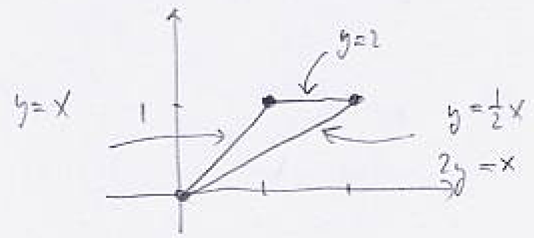
Calculus 3 — Exam 4 **MAT 309, Fall 2012 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

1. (18pts) Find $\iint_D x + y\,dA$ if D is the triangle with vertices $(0,0)$, $(1,1)$, $(2,1)$. Sketch the region of integration.



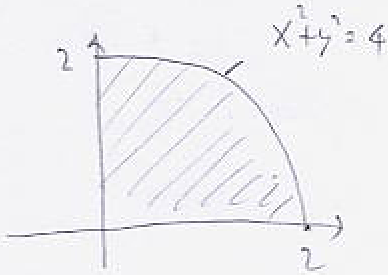
$$\iint_D x+y\,dA = \left[\begin{array}{l}\text{view region}\\ \text{as horizontally simple}\end{array}\right]$$

$$= \int_0^1\int_y^{2y} x+y\,dx\,dy = \int_0^1 \left(\frac{x^2}{2}\Big|_y^{2y}\right) + y(2y-y)\,dy$$

$$= \int_0^1 \frac{1}{2}(4y^2-y^2)+y^2\,dy = \int_0^1 \frac{5}{2}y^2\,dy$$

$$= \frac{5}{2}\cdot\frac{1}{3}y^3\Big|_0^1 = \frac{5}{6}(1-0) = \frac{5}{6}$$

2. (18pts) Evaluate $\int_0^2\int_0^{\sqrt{4-x^2}} (4-y^2)^{\frac{3}{2}}\,dy\,dx$ by changing the order of integration. Sketch the region of integration.



$$\iint_D (4-y^2)^{\frac{3}{2}}\,dA = \int_0^2\int_0^{\sqrt{4-y^2}} (4-y^2)^{\frac{3}{2}}\,dx\,dy$$

$$= \int_0^2 (4-y^2)^{\frac{3}{2}}\left(\sqrt{4-y^2}-0\right)dy = \int_0^2 (4-y^2)^2\,dy$$

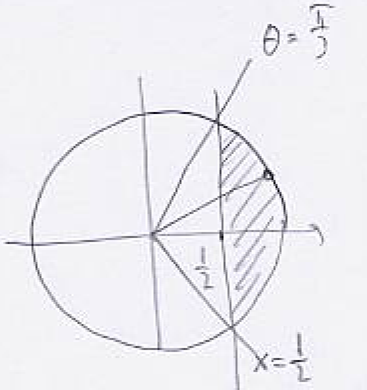
$$= \int_0^2 16-8y^2+y^4\,dy$$

$$= 16\cdot 2 - 8\frac{1}{3}y^3\Big|_0^2 + \frac{1}{5}y^5\Big|_0^2 = 32-\frac{8}{3}\cdot(8-0)+\frac{1}{5}(32-0)$$

$$= 32-\frac{64}{3}+\frac{32}{5} = \frac{32}{3}+\frac{32}{5} = \frac{160+96}{15} = \frac{256}{15}$$

$\frac{96}{3}$

3. (16pts) Use polar coordinates to find the area of the region that is inside the unit circle and to the right of the line $x = \frac{1}{2}$. Sketch the region of integration first.



$r\cos\theta = \frac{1}{2}$

$r = \frac{1}{2\cos\theta}$

$\cos\theta = \frac{1}{2} \quad \theta = -\frac{\pi}{3}, \frac{\pi}{3}$

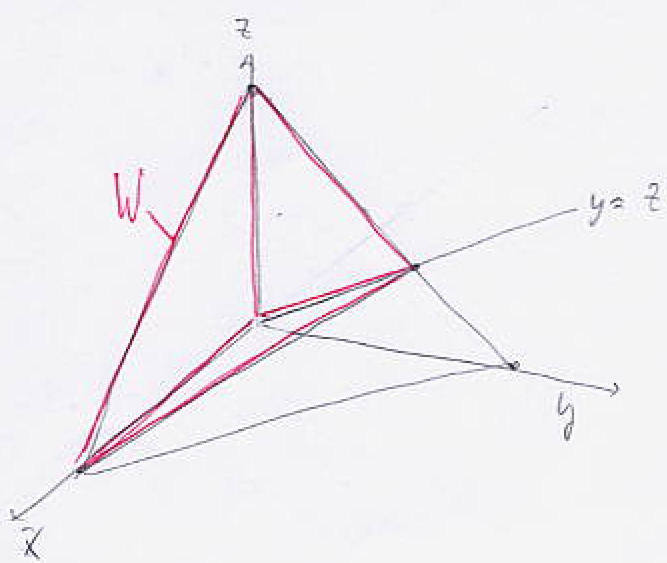
$$A = \iint_D 1\, dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{1}{2\cos\theta}}^{1} 1 \cdot r\, dr\, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left(\frac{r^2}{2} \Big|_{\frac{1}{2\cos\theta}}^{1} \right) d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} 1 - \frac{1}{4\cos^2\theta}\, d\theta \qquad = \frac{1}{4}\sec^2\theta$$

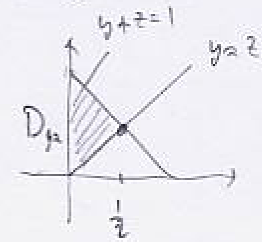
$$= \frac{1}{2}\left(\frac{2\pi}{3} - \frac{1}{4}\tan\theta \Big|_{-\pi/3}^{\pi/3} \right) = \frac{1}{2}\left(\frac{2\pi}{3} - \left(\frac{1}{4}\sqrt{3} - \frac{1}{4}(-\sqrt{3}) \right) \right)$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} - \frac{1}{2}\sqrt{3} \right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

4. (16pts) Sketch the region W that is in the first octant $(x, y, z \geq 0)$, above the plane $y = z$ and below the plane $x + y + z = 1$. Then write the two iterated triple integrals that stand for $\iiint_W f\, dV$ which end in $dx\, dz\, dy$, and $dz\, dy\, dx$.

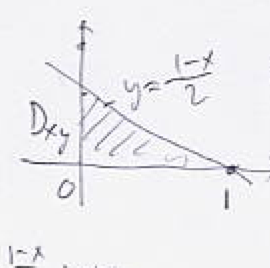


$dx\,dy\,dz$:
proj. to yz-plane



$$\iint_{D_{yz}} \int_0^{1-y-z} f\, dx\, dA = \int_0^{\frac{1}{2}} \int_y^{1-y} \int_0^{1-y-z} f\, dx\, dz\, dy$$

$dz\,dy\,dx$:
proj. to xy-plane



$y = z$
$x + y + z = 1$

$x + 2y = 1$

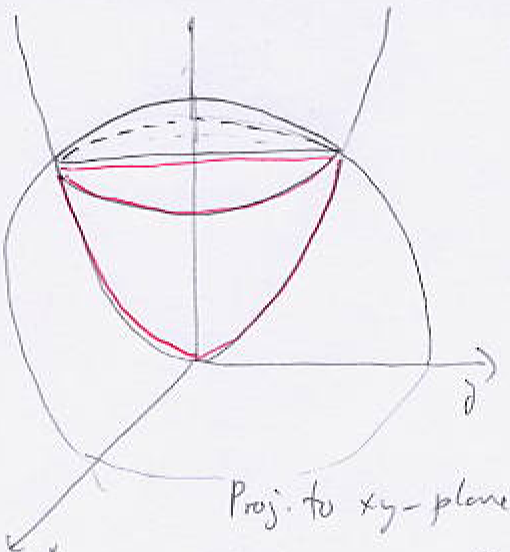
$y = \frac{1-x}{2}$

$x + y + z = 1$

$\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$ $\quad x, y, z$-intercepts all 1

$$\iint_{D_{xy}} \int_y^{1-x-y} f\, dz\, dA = \int_0^1 \int_0^{\frac{1-x}{2}} \int_y^{1-x-y} f \quad dz\, dy\, dx$$

5. (16pts) Use cylindrical coordinates to set up $\iiint_W \frac{x+y+z}{x^2+y^2}\,dV$ where W is the part of the region above the paraboloid $z = x^2+y^2$ and below the sphere $x^2+y^2+z^2 = 12$ where $x \geq 0$. Sketch the region of integration. Do not evaluate the integral.

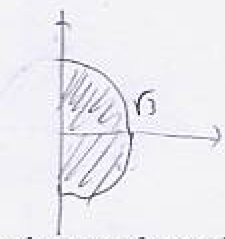


$$\iint_D \int_{x^2+y^2}^{\sqrt{12-(x^2+y^2)}} \frac{x+y+z}{x^2+y^2}\,dz\,dA$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{3}}\int_{r^2}^{\sqrt{12-r^2}} \frac{r\cos\theta + r\sin\theta + z}{r^2}\cdot r\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{3}}\int_{r^2}^{\sqrt{12-r^2}} \frac{r\cos\theta + r\sin\theta + z}{r}\,dz\,dr\,d\theta$$

Proj. to xy-plane:



$x^2+y^2+z^2 = 12$

$z = x^2+y^2$

$z^2+z = 12$

$z^2+z-12 = 0$

$(z+4)(z-3) = 0$

$z = -4, 3$

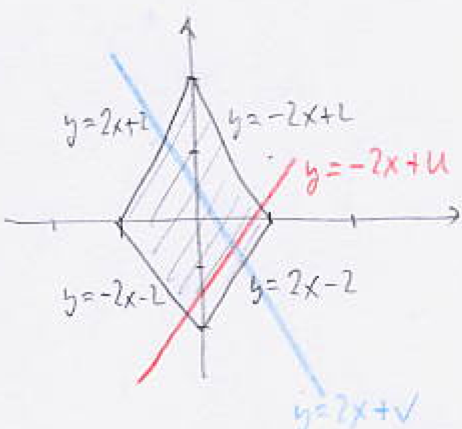
$\uparrow$ not ans.

$z = 3$

$x^2+y^2 = 3$

radius $= \sqrt{3}$

6. (16pts) Use change of variables to find the integral $\iint_D (2x+y)^2\,dA$ if D is the rhombus bounded by $y = 2x+2$, $y = 2x-2$, $y = -2x+2$, $y = -2x-2$. Sketch the region D.



$$\begin{cases} y = -2x+u \\ y = 2x+v \end{cases} \Rightarrow 2y = u+v, \quad y = \tfrac{1}{2}(u+v)$$

$$x = \tfrac{1}{2}(y-v) = \tfrac{1}{2}\left(\tfrac{1}{2}(u-v)\right) = \tfrac{1}{4}(u-v)$$

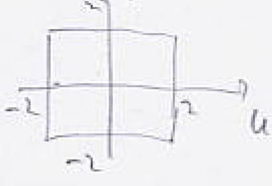
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{8} - \left(-\frac{1}{8}\right) = \frac{1}{4}$$

$-2 \leq u \leq 2$

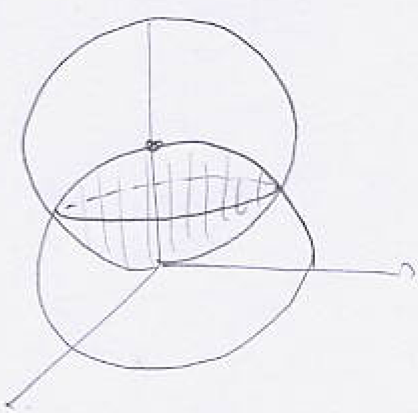
$-2 \leq v \leq 2$

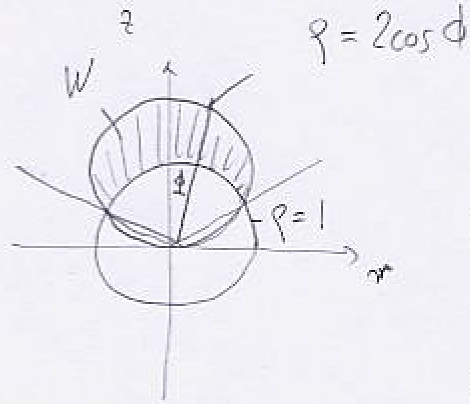
$$\iint_D \underbrace{(2x+y)}_{=u}{}^2\,dA = \int_{-2}^{2}\int_{-2}^{2} u^2 \left|\tfrac{1}{4}\right| dv\,du$$

$$= \frac{1}{4}\int_{-2}^{2} u^2\,du \int_{-2}^{2} dv = \frac{1}{4}\cdot 4\left(\frac{u^3}{3}\Big|_{-2}^{2}\right) = \frac{1}{3}(8-(-8)) = \frac{16}{3}$$



Bonus. (10pts) The intersection of balls $x^2 + y^2 + z^2 \leq 1$ and $x^2 + y^2 + (z-1)^2 \leq 1$ is a lens-shaped region. Find its volume by doing the following:
a) Use spherical coordinates to find the volume of the region inside $x^2 + y^2 + (z-1)^2 \leq 1$ that is outside of $x^2 + y^2 + z^2 \leq 1$.
b) Find the volume of the lens using your result from a). Recall that the volume of a ball of radius R is $\frac{4}{3}\pi R^3$.





$\begin{cases}\rho=1\\ \rho=2\cos\phi\end{cases}$

$2\cos\phi=1$

$\cos\phi=\frac{1}{2}$

$\phi=\frac{\pi}{3}, -\frac{\pi}{3}$ ← not in the interval $[0,\pi]$

$$\text{Volume of } W = \iiint_W 1\,dV = \int_0^{2\pi}\underbrace{\int_0^{\frac{\pi}{3}}\int_1^{2\cos\phi} 1\,\rho^2\sin\phi\,d\rho\,d\phi}_{\text{constant for }\theta}\,d\theta$$

$$= 2\pi\cdot\int_0^{\frac{\pi}{3}}\sin\phi\left(\frac{\rho^3}{3}\right)\Big|_1^{2\cos\phi}d\phi = \frac{2\pi}{3}\int_0^{\pi/3}\sin\phi\left(8\cos^3\phi-1\right)d\phi$$

$$= \frac{2\pi}{3}\left(8\int_0^{\pi/3}\sin\phi\cos^3\phi\,d\phi-\int_0^{\pi/3}\sin\phi\,d\phi\right)=\frac{2\pi}{3}\left(8\left(-\frac{\cos^4\phi}{4}\right)\Big|_0^{\pi/3}+\cos\phi\Big|_0^{\pi/3}\right)$$

$$= \frac{2\pi}{3}\left(-\frac{8}{4}\left(\left(\frac{1}{2}\right)^4-1\right)+\frac{1}{2}-1\right)=\frac{2\pi}{3}\left(\frac{15}{8}-\frac{1}{2}\right)=\frac{2\pi}{3}\cdot\frac{11}{8}=\frac{11\pi}{12}$$

b) Volume of lens = volume of upper sphere − volume of W

$$= \frac{4}{3}\cdot\pi\cdot 1-\frac{11\pi}{12}=\frac{16\pi-11\pi}{12}=\frac{5\pi}{12}$$