Calculus 3 — Exam 3
MAT 309, Fall 2012 — D. Ivanšić

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1. (12pts) Find the equation of the tangent plane to the surface $x^2 - \frac{y^2}{4} - \frac{z^2}{9} = 1$ at the point $\left(2, \sqrt{2}, 3\sqrt{\frac{5}{2}}\right)$. Simplify the equation to standard form.

$f(x,y,z) = x^2 - \frac{y^2}{4} - \frac{z^2}{9}$

$\nabla f = \langle 2x, -\frac{1}{2}y, -\frac{2}{9}z \rangle$

$\nabla f(2, \sqrt{2}, 3\sqrt{\frac{5}{2}}) = \langle 4, -\frac{\sqrt{2}}{2}, -\frac{2}{9}3\sqrt{\frac{5}{2}} \rangle = \langle 4, -\frac{\sqrt{2}}{2}, -\frac{2}{3}\sqrt{\frac{5}{2}} \rangle$

Equation of plane:

$4x - \frac{\sqrt{2}}{2}y - \frac{2}{3}\sqrt{\frac{5}{2}}z = 8 - 1 - 2\cdot\frac{5}{2}$

$4x - \frac{\sqrt{2}}{2}y - \frac{2\sqrt{5}}{3\sqrt{2}}z = 2 \quad | \cdot 3\sqrt{2}$

$12\sqrt{2}x - 3y - 2\sqrt{5}z = 6\sqrt{2}$

2. (18pts) Let $f(x, y) = x^2 - y^2$.

a) Find the directional derivative of f at the point $(3, 2)$ in the direction of $\mathbf{v} = \langle 1, -5 \rangle$.

b) At the point $(3, 2)$, in which direction does the function increase the fastest? Decrease the fastest?

c) Find $\frac{d}{dt} f(\mathbf{r}(t))$ for the path $\mathbf{r}(t) = \langle 4 - 3t, t^2 - 2t \rangle$ when $t = 1$.

a) $\nabla f = \langle 2x, -2y \rangle$

$D_{\vec{u}} f = \nabla f(3,2) \cdot \frac{1}{\|\vec{v}\|} \vec{v}$

$= \langle 6, -4 \rangle \cdot \frac{1}{\sqrt{1+25}} \langle 1, -5 \rangle$

$= \frac{1}{\sqrt{26}} \cdot 26 = \sqrt{26}$

b) Fastest increase in direction of $\nabla f = \langle 6, -4 \rangle$

Fastest decrease in direction of $-\nabla f = \langle -6, 4 \rangle$

c) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$\vec{r}\,'(t) = \langle -3, 2t-2 \rangle$

$\vec{r}(1) = \langle 1, -1 \rangle \quad \vec{r}\,'(1) = \langle -3, 0 \rangle$

$\frac{d}{dt} f(\vec{r}(t)) \Big|_{t=1} = \nabla f(1,-1) \cdot \langle -3, 0 \rangle$

$= \langle 2, 2 \rangle \cdot \langle -3, 0 \rangle$

$= -6$

3. (20pts) Let $f(x,y) = \dfrac{\ln x}{\ln x + \ln y}$, $x = e^u \cos v$, $y = e^u \sin v$. Use the chain rule to find $\dfrac{\partial f}{\partial v}$ when $u = 3$, $v = \dfrac{\pi}{6}$.

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\frac{1}{x}(\ln x + \ln y) - \ln x \cdot \frac{1}{x}}{(\ln x + \ln y)^2} \cdot e^u(-\sin v) + -\frac{\ln x}{(\ln x + \ln y)^2}\cdot\frac{1}{y}\cdot e^u \cos v$$

$$= -\frac{\ln y}{x(\ln x + \ln y)^2} e^u \sin v - \frac{\ln x}{y(\ln x + \ln y)^2} e^u \cos v$$

When $u = 3$, $v = \frac{\pi}{6}$, $x = e^3 \cdot \frac{\sqrt{3}}{2}$, $y = \frac{e^3}{2}$

$$\left.\frac{\partial f}{\partial v}\right|_{(3,\frac{\pi}{6})} = -\frac{\ln\left(\frac{e^3}{2}\right)\cdot e^3 \cdot \frac{1}{2}}{e^3 \frac{\sqrt{3}}{2}\left(\ln\left(e^3 \frac{\sqrt{3}}{2}\right) + \ln\left(\frac{e^3}{2}\right)\right)^2} - \frac{\ln\left(e^3 \cdot \frac{\sqrt{3}}{2}\right)\cdot e^3 \cdot \frac{\sqrt{3}}{2}}{\frac{e^3}{2}\left(\ln\left(e^3 \frac{\sqrt{3}}{2}\right) + \ln\left(\frac{e^3}{2}\right)\right)^2} = \left[\ln e^3 = 3\right]$$

$$= -\frac{(3 - \ln 2)}{\sqrt{3}(3 + \ln\sqrt{3} - \ln 2 + 3 - \ln 2)^2} - \frac{(3 + \ln\sqrt{3} - \ln 2)\sqrt{3}}{(3 + \ln\sqrt{3} - \ln 2 + 3 - \ln 2)^2} = \boxed{-\frac{12 - 4\ln 2 + 3\ln\sqrt{3}}{\sqrt{3}(6 + \ln\sqrt{3} - 2\ln 2)^2}}$$

In numerator: $3 - \ln 2 + 9 + 3\ln\sqrt{3} - 3\ln 2$

4. (14pts) In an improbable scenario, you find yourself on a desert island with no technology and have to estimate $\dfrac{\sqrt{20.2}}{\sqrt{4.99}}$. (Or, in a more likely one, you find yourself in test-taking environment with no calculators allowed.) Use linearization to estimate this number, and compare it to the calculator result of 2.011988.

$$f(x,y) = \frac{\sqrt{x}}{\sqrt{y}} \qquad f_x = \frac{1}{2\sqrt{x}\sqrt{y}} = \frac{1}{2\sqrt{xy}} \qquad f_y = \sqrt{x}\left(-\tfrac{1}{2}\right)y^{\frac{3}{2}} = -\frac{\sqrt{x}}{2(\sqrt{y})^3}$$

$$f(20,5) = \frac{\sqrt{20}}{\sqrt{5}} = \sqrt{4} = 2$$

$$f(a + \Delta x, b + \Delta y) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

$$f(20.2, 4.99) \approx 2 + \frac{1}{2\sqrt{20 \cdot 5}}\cdot 0.2 - \frac{\sqrt{20}}{2\sqrt{5}^3}(-0.01)$$

$$= 2 + \frac{0.2}{20} + \frac{2\sqrt{5}}{2\cdot 5\sqrt{5}}\,0.01 = 2 + 0.01 + \overbrace{0.2\cdot 0.01}^{0.002}$$

$$\frac{0.2}{20} = \frac{0.1}{10} \qquad \frac{2\sqrt{5}}{2\cdot 5\sqrt{5}} = \frac{1}{5} = 0.2$$

$= 2.012$ about 0.000012 away from actual value

5. (14pts) Using implicit differentiation, find $\dfrac{\partial z}{\partial y}$ at the point $(-1, 2, 1)$, if $x^2y+y^2z+z^2x = 5$.

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \qquad F(x,y,z) = x^2y + y^2z + z^2x$$

$$F_y = x^2 + 2yz$$

$$F_z = y^2 + 2xz$$

$$\left.\frac{\partial z}{\partial y}\right|_{(-1,2,1)} = -\frac{x^2+2yz}{y^2+2xz} = -\frac{1+2\cdot2\cdot1}{4+2\cdot(-1)\cdot1} = -\frac{5}{2}$$

6. (22pts) Find and classify the local extremes for $f(x,y) = \frac{2}{3}x^3 + 2xy - 2y^2 - 10x$.

$$f_x = 2x^2 + 2y - 10$$

$$f_y = 2x - 4y$$

$$D = \begin{vmatrix} 4x & 2 \\ 2 & -4 \end{vmatrix}$$

$$\begin{cases} x^2 + y - 5 = 0 \\ x - 2y = 0 \Rightarrow x = 2y, \text{ put in first eq.} \end{cases}$$

$$D\left(-\tfrac{5}{2}, -\tfrac{5}{4}\right) = \begin{vmatrix} -10 & 2 \\ 2 & -4 \end{vmatrix} = 36 > 0$$

Local max since $f_{xx} < 0$

$$(2y)^2 + y - 5 = 0$$

$$4y^2 + y - 5 = 0$$

$$D(2,1) = \begin{vmatrix} 8 & 2 \\ 2 & -4 \end{vmatrix} = -36 < 0$$

$$y = \frac{-1 \pm \sqrt{1 - 4\cdot4(-5)}}{2\cdot4} = \frac{-1\pm\sqrt{81}}{8}$$

So $(2,1)$ is a saddle point.

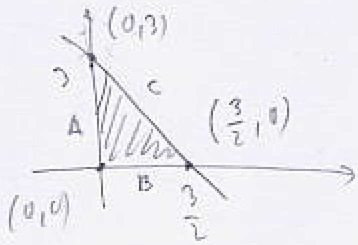
$$= \frac{-1\pm9}{8} = -\frac{5}{4}, 1$$

Then $x = -\frac{5}{2}, 2$

Candidates: $\left(-\frac{5}{2}, -\frac{5}{4}\right)$, $(2,1)$

local max   saddle

Bonus (10pts) Find the absolute maximum and minimum of $f(x, y) = x^2 + y^2 - 6x - 4y$ on the domain $x \geq 0$, $y \geq 0$, $y \leq -2x + 3$.



$f_x = 2x - 6$ $\qquad$ $\begin{cases} 2x-6=0 \\ 2y-4=0 \end{cases}$ $\quad$ $x=3$, $y=4$ $\qquad$ Crit. pt. $(3,4)$ not in region

$f_y = 2y - 4$

Boundary:

A $\quad$ $x=0$, $y=t$ $\quad$ $t \in [0,3]$ $\quad$ $f(0,t) = t^2 - 4t$

Crit. pt. $2t-4=0$, $t=2 \to (0,2)$

B $\quad$ $x=t$, $y=0$ $\quad$ $t \in [0, \frac{3}{2}]$ $\quad$ $f(t,0) = t^2 - 6t$

Crit. pts $2t-6=0$, $t=3$, not in interval

C $\quad$ $x=t$, $y=-2t+3$ $\quad$ $t \in [0, \frac{3}{2}]$ $\quad$ $f(t, -2t+3) = t^2 + (-2t+3)^2 - 6t - 4(-2t+3)$

$= t^2 + 4t^2 - 12t + 9 - 6t + 8t - 12$

$= 5t^2 - 10t - 3$

Crit pts: $10t - 10 = 0$, $t = 1 \rightsquigarrow (1,1)$

	Candidates	$f(x,y)$	
corners	$(0,0)$	0	max
	$(0,3)$	$9-12=-3$	
	$(\frac{3}{2},0)$	$\frac{9}{4}-9=-\frac{27}{4}$	
	$(0,2)$	$4-8=-4$	
	$(1,1)$	$1+1-6-4=-8$	min