Calculus 3 — Exam 2
MAT 309, Fall 2012 — D. Ivanšić

Name: Saul Ocean

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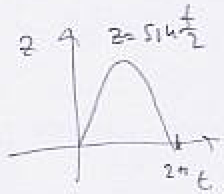
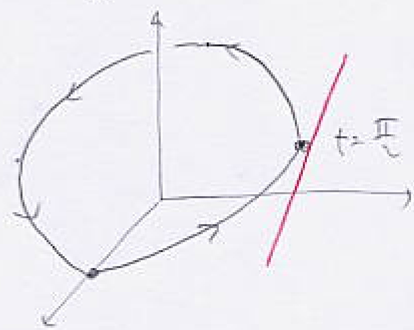
1. (14pts) A curve is given by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin \frac{t}{2} \rangle$, $t \in [0, 2\pi]$.
a) Sketch this curve.
b) Find the parametric equation of the tangent line to the curve at time $t = \frac{\pi}{2}$ and draw this tangent line on your sketch.

a) In the xy plane, there is rotation.



z-coordinate goes from 0 to 1 and back



b) $\vec{r}\left(\frac{\pi}{2}\right) = \left\langle \cos\frac{\pi}{2}, \sin\frac{\pi}{2}, \sin\frac{\pi}{4} \right\rangle = \left\langle 0, 1, \frac{\sqrt{2}}{2} \right\rangle$

$\vec{r}\,'(t) = \left\langle -\sin t, \cos t, \frac{1}{2}\cos\frac{t}{2} \right\rangle$

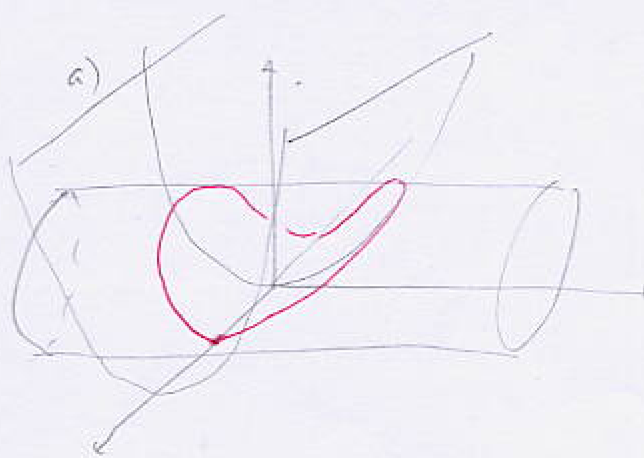
$\vec{r}\,'\left(\frac{\pi}{2}\right) = \left\langle -\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, \frac{1}{2}\cos\frac{\pi}{4} \right\rangle = \left\langle -1, 0, \frac{\sqrt{2}}{4} \right\rangle$

Param. equations of tangent line:

$x(t) = -t$
$y(t) = 1$
$z(t) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}t$

2. (16pts) Consider the curve C that is the intersection of the cylinder $x^2 + z^2 = 4$ with the parabolic cylinder $z = y^2$.
a) Sketch a picture.
b) Parametrize each of the two parts of the curve corresponding to $x \geq 0$ and $x \leq 0$, taking y as the parameter.
c) What interval does the parameter run through to get each of the two parts?

a)



b) $x^2 + z^2 = 4$
$z = y^2$
$x^2 = 4 - z^2 = 4 - y^4$
$x = \pm\sqrt{4 - y^4}$

Thus

$x = \sqrt{4 - t^4}$ $\quad$ $x = -\sqrt{4 - t^4}$
$y = t$ $\quad$ $y = t$
$z = t^2$ $\quad$ $z = t^2$

c) Must have $4 - t^4 \geq 0$

$t^4 \leq 4$

$|t| \leq \sqrt[4]{4}$ $\quad$ so $-\sqrt{2} \leq t \leq \sqrt{2}$

$|t| \leq \sqrt{2}$

3 5. (22pts) Consider the function $f(x, y) = \dfrac{y}{x^2}$ for $x > 0$, y any.

a) Identify and draw vertical traces for this function.

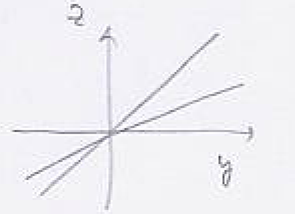
b) Using a), draw the graph of the function.

c) Draw a rough contour map for the function, with contour interval $\frac{1}{2}$, going from $c = -\frac{3}{2}$ to $c = \frac{3}{2}$.

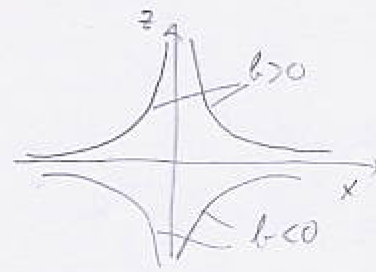
d) By looking at the contour map, indicate a path on which we could move from $(\sqrt{2}, 1)$ in order to increase the value of the function to 1.

Fix

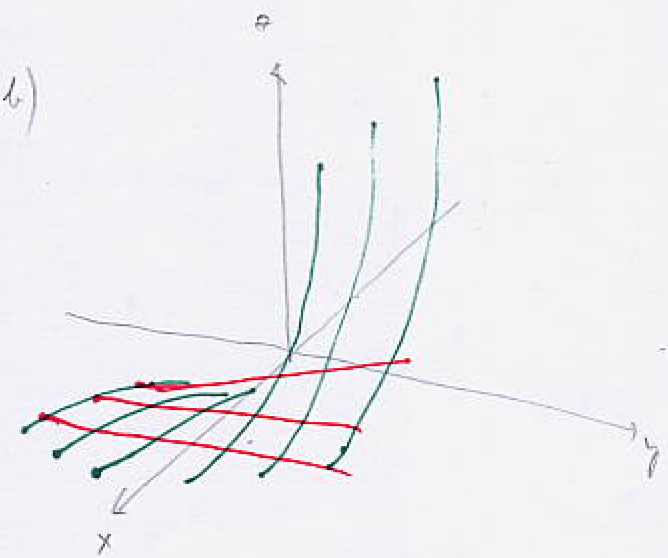
a) $x = a$ $\quad z = \dfrac{y}{a^2} = \dfrac{1}{a^2} y$ lines with positive slope



$y = b$ $\quad z = \dfrac{b}{x^2} = b \cdot \dfrac{1}{x^2}$



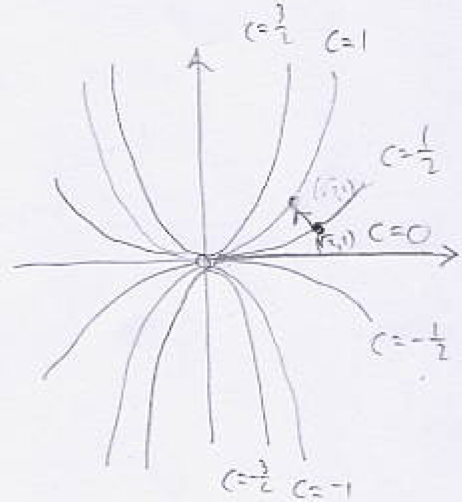
b)



c) $\dfrac{y}{x^2} = c$

$y = cx^2$

parabolas.



d) $f(\sqrt{2}, 1) = \frac{1}{2}$ see picture

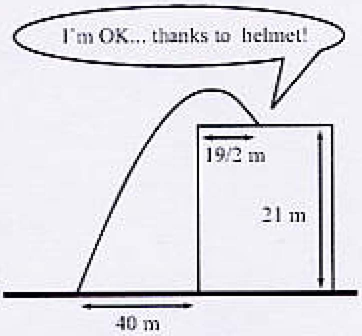
4. (16pts) Find the length of the curve with the parametrization $\mathbf{r}(t) = \left\langle t^{\frac{3}{2}}, 5\sin t, 5\cos t \right\rangle$, $t \in [0, 4\pi]$.

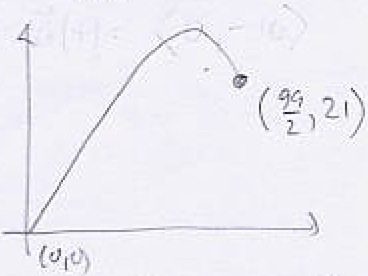
$\vec{r}\,'(t) = \langle \frac{3}{2}t^{\frac{1}{2}}, 5\cos t, -5\sin t \rangle$

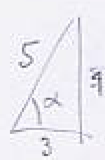
$\|\vec{r}\,'(t)\| = \sqrt{\frac{9}{4}t + 25\cos^2 t + 25\sin^2 t} = \sqrt{\frac{9}{4}t + 25}$

$\int_0^{4\pi} \|\vec{r}\,'(t)\| dt = \int_0^{4\pi} \left(\frac{9}{4}t + 25\right)^{\frac{1}{2}} = \left.\frac{\left(\frac{9}{4}t+25\right)^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{9}{4}}\right|_0^{4\pi} = \frac{27}{8}\left((9\pi+25)^{\frac{3}{2}} - 125\right)$

5 **3.** (20pts) Acting on a dare, your favorite physics professor Dr. ________ (insert name) launches himself from 40 meters away from base of Faculty Hall (height 21 meters) and lands on its roof $\frac{19}{2}$ meters away from the edge. (See picture.) The angle α of launch was such that $\cos\alpha = \frac{3}{5}$. Assume $g = 10$ for simplicity.
a) Find his position at time t. The expression will have an unknown initial speed v_0 in it.
b) Now find v_0 and insert it in the position function.







$\vec{r}(0) = \vec{0}$, $\vec{v}(0) = \langle v_0\cos\alpha, v_0\sin\alpha \rangle = \langle v_0\cdot\frac{3}{5}, v_0\frac{4}{5}\rangle$

a) $\vec{a}(t) = \langle 0, -10 \rangle$

$\vec{v}(t) = \langle 0, -10t \rangle + \vec{C}$

$\langle v_0\frac{3}{5}, v_0\frac{4}{5} \rangle = \vec{v}(0) = \vec{C}$

$\vec{v}(t) = \langle v_0\frac{3}{5}, v_0\frac{4}{5} - 10t \rangle$

$\vec{r}(t) = \langle v_0\frac{3}{5}t, v_0\frac{4}{5}t - 5t^2 \rangle + \vec{D}$

$\vec{0} = \vec{r}(0) = 0 + \vec{D}$ so $\vec{D} = 0$

b) For some t, $\vec{r}(t) = \langle \frac{99}{2}, 21 \rangle$

$v_0\frac{3}{5}t = \frac{99}{2} \Rightarrow v_0 t = \frac{5\cdot 99}{3\cdot 2} = \frac{165}{2}$

$v_0\frac{4}{5}t - 5t^2 = 21 \Rightarrow \frac{4}{5}\cdot\frac{165}{2} - 5t^2 = 21$

$66 - 5t^2 = 21$ $\qquad v_0\cdot 3 = \frac{165}{2}$

$5t^2 = 45$ $\qquad$ so $v_0 = \frac{165}{3\cdot 2} = \frac{55}{2}$ m/s

$t^2 = 9$

$t = 3, -3$ $\qquad \vec{r}(t) = \langle \frac{33}{2}t, 22t - 5t^2 \rangle$

6. (12pts) Determine and sketch the domain of the function $f(x,y) = \dfrac{\sqrt{x-y-5}}{x+y}$.

Must have

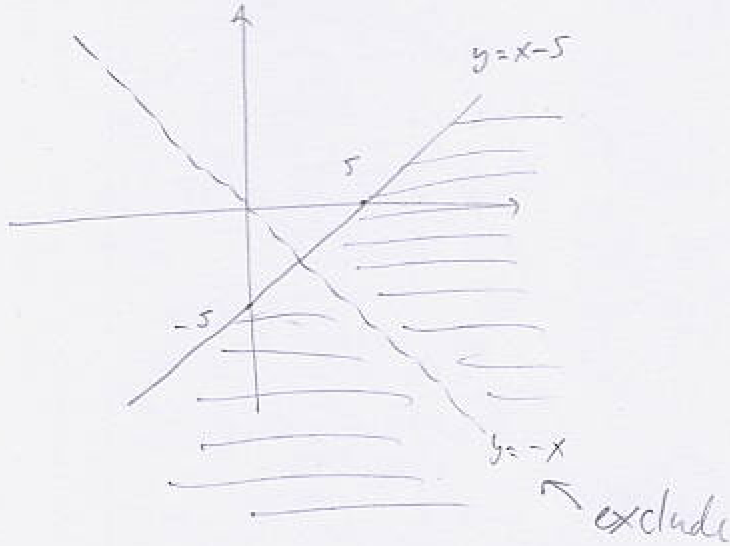
$x-y-5 \geq 0$

$y \leq x-5$

Can't have

$x+y=0$

$y=-x$



Bonus (10pts) Use coordinates to prove the formula $(\mathbf{u}(t)\times\mathbf{v}(t))' = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$ for any two vector functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

Let $\vec{u}(t) = \langle x_1(t), y_1(t), z_1(t)\rangle$, $\vec{v}(t) = \langle x_2(t), y_2(t), z_2(t)\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1z_2 - y_2z_1,\ x_2z_1 - x_1z_2,\ x_1y_2 - x_2y_1\rangle$$

$$(\vec{u}\times\vec{v})' = \langle y_1'z_2 + y_1z_2' - y_2'z_1 - y_2z_1',\ x_2'z_1 + x_2z_1' - x_1'z_2 - x_1z_2',\ x_1'y_2 + x_1y_2' - x_2'y_1 - x_2y_1'\rangle$$

$$\vec{u}'\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1' & y_1' & z_1' \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1'z_2 - y_2z_1',\ x_2z_1' - x_1'z_2,\ x_1'y_2 - x_2y_1'\rangle$$

are equal

$$\vec{u}\times\vec{v}' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2' & y_2' & z_2' \end{vmatrix} = \langle y_1z_2' - y_2'z_1,\ x_2'z_1 - x_1z_2',\ x_1y_2' - x_2'y_1\rangle$$

$$\vec{u}'\times\vec{v} + \vec{u}\times\vec{v}' = \langle y_1'z_2 + y_1z_2' - y_2'z_1 - y_2z_1',\ x_2'z_1 + x_2z_1' - x_1'z_2 - x_1z_2',\ x_1'y_2 + x_1y_2' - x_2'y_1 - x_2y_1'\rangle$$