Calculus 3 — Exam 1
MAT 309, Fall 2012 — D. Ivanšić

Name: Saul Ocean

Show all your work!

1. (18pts) Let $\mathbf{u} = \langle 3, 1, -3 \rangle$ and $\mathbf{v} = \langle 0, 2, -1 \rangle$.
a) Calculate $2\mathbf{u}$, $4\mathbf{u} - 3\mathbf{v}$, and $||\mathbf{u}||$.
b) Find the unit vector in direction of $\mathbf{v}$.
c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

a) $2\vec{u} = \langle 6, 2, -6 \rangle$

$4\vec{u} - 3\vec{v} = \langle 12, 4, -12 \rangle - \langle 0, 6, -3 \rangle = \langle 12, -2, -9 \rangle$

$||\vec{u}|| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19}$

b) $\vec{e}_v = \frac{1}{\sqrt{0^2 + 2^2 + (-1)^2}} \langle 0, 2, -1 \rangle = \frac{1}{\sqrt{5}} \langle 0, 2, -1 \rangle$

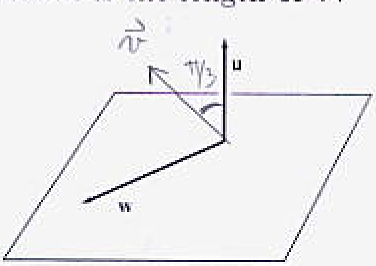
c) $\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \vec{v} = \frac{0 + 2 + 3}{5} \langle 0, 2, -1 \rangle = \langle 0, 2, -1 \rangle$

2. (4pts) Do the coordinate systems given by the sets of vectors below (in order listed) satisfy the right hand rule?

$\{\mathbf{j}, \mathbf{k}, \mathbf{i}\}$ Yes $(\vec{j} \times \vec{k} = \vec{i})$

$\{\mathbf{i}, \mathbf{k}, \mathbf{j}\}$ No $(\vec{i} \times \vec{k} = -\vec{j})$

3. (10pts) Vector $\mathbf{u}$ is perpendicular to the plane containing $\mathbf{w}$ (picture). Their lengths are $||\mathbf{u}|| = 3$ and $||\mathbf{w}|| = 5$. Draw a vector $\mathbf{v}$ whose angle with $\mathbf{u}$ is $\pi/3$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. What is the length of $\mathbf{v}$?



$||\vec{u}|| \, ||\vec{v}|| \sin \frac{\pi}{3} = ||\vec{u} \times \vec{v}||$

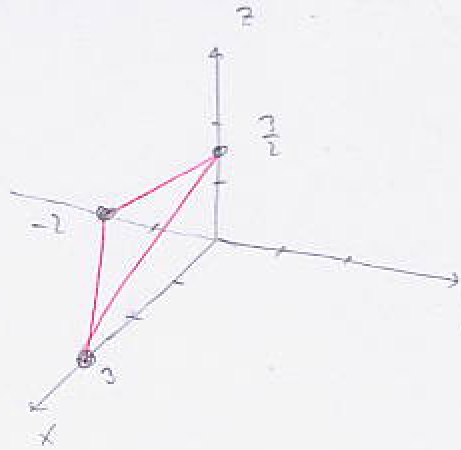
$3 \cdot ||\vec{v}|| \frac{\sqrt{3}}{2} = 5$

$||\vec{v}|| = \frac{10}{3\sqrt{3}}$

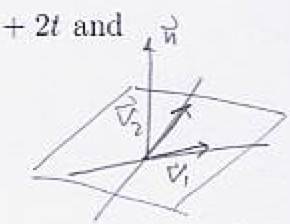
($\vec{v}$ is in plane perpendicular to $\vec{u}$)

4. (12pts) Find the points of intersection of the plane $2x - 3y + 4z = 6$ with the x-, y- and z-axes and use this information to sketch the plane in a coordinate system.

$y=0$	$x=0$	$x=0$
$z=0$	$z=0$	$y=0$
$2x=6$	$-3y=6$	$4z=6$
$x=3$	$y=-2$	$z=\frac{3}{2}$



5. (20pts) Two lines are given parametrically: $x = -10 - 3t$, $y = 5 + t$, $z = 10 + 2t$ and $x = -5 + 2s$, $y = -2 + 2s$, $z = 6 - s$.
a) Show that these lines intersect by finding the point of intersection.
b) Find the equation of the plane spanned by these two lines.



a) $\begin{cases} -10-3t = -5+2s \\ 5+t = -2+2s \\ 10+2t = 6-s \end{cases}$

$\begin{cases} 3t+2s = -5 \\ t-2s = -7 \\ 2t+s = -4 \end{cases}$ add $\quad 4t = -12$, $t = -3$

So $s = \frac{-7-t}{-2} = 2$

$2\cdot(-3)+2 = -4$ satisfy third equation

so system does have a solution

$t=-3$, $s=2$

Point is $(-10-3(-3), 5+(-3), 10+2\cdot(-3))$

$= (-1, 2, 4)$

b) $\vec{n} = \vec{v}_1 \times \vec{v}_2$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 2 & 2 & -1 \end{vmatrix} = (-1-4)\vec{i} - (3-4)\vec{j} + (-6-2)\vec{k}$

$= -5\vec{i} + \vec{j} - 8\vec{k}$

May take $5\vec{i} - \vec{j} + 8\vec{k} = \vec{n}$

Equation of plane:

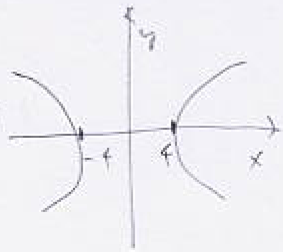
$5x - y + 8z = 5\cdot(-1) - 2 + 8\cdot 4$

$5x - y + 8z = 25$

6. (16pts) This problem is about the surface $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

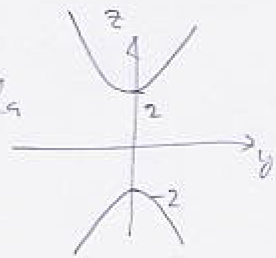
a) $z = 0$

$\left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$ hyperbola



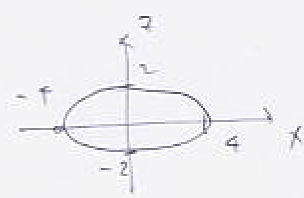
$x = 0$

$\left(\frac{z}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$ hyperbola

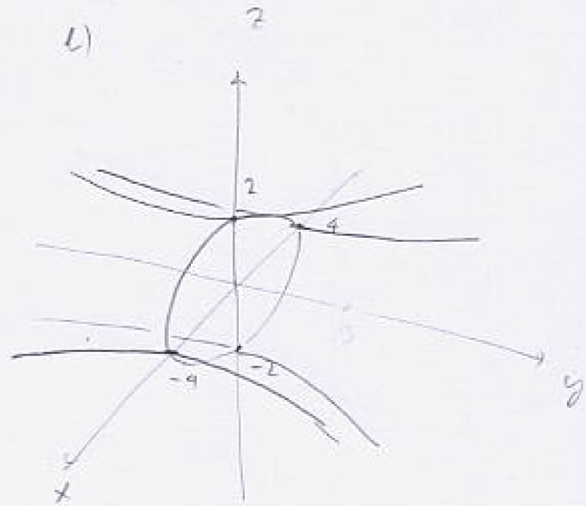


$y = 0$

$\left(\frac{x}{4}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$ ellipse



b)

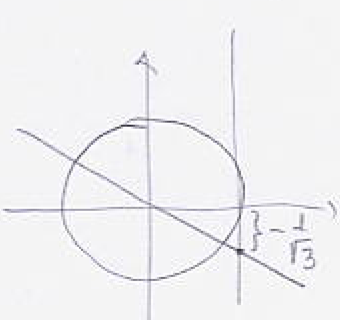


Hyperboloid of one sheet around y-axis

7. (10pts) Find the cylindrical coordinates of the point whose cartesian coordinates are $(-2\sqrt{3}, 2, 4)$.

$\tan\theta = \frac{y}{x} = \frac{2}{-2\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$r = \sqrt{(-2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4$



$\theta = -\frac{\pi}{6}, \frac{5\pi}{6}$

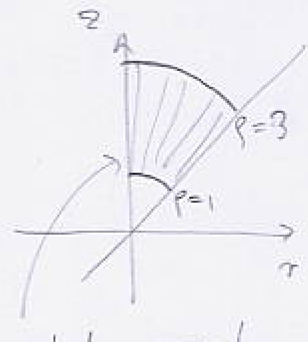
since (x, y) is in 2nd quadrant,

$\theta = \frac{5\pi}{6}$

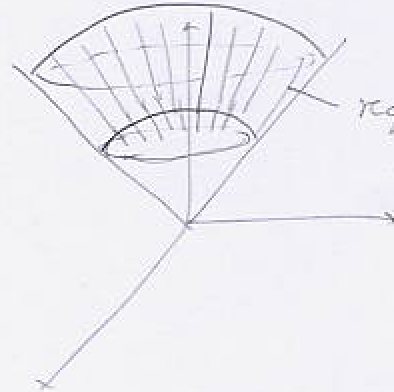
$(r, \theta, z) = \left(4, \frac{5\pi}{6}, 4\right)$

8. (10pts) Sketch the following set of points given in spherical coordinates:

$0 \leq \phi \leq \frac{\pi}{4}, 1 \leq \rho \leq 3$



rotate around
z axis



region between spheres
of radii 1, 3,
within a cone.

Bonus (10pts) Find a plane that contains the x-axis and has angle $\pi/3$ with the plane $x + y = 4$. How many such planes are there? *(Hints: recall that angle between planes is the angle between their normal vectors. Look for a unit normal vector.)*

Let $\vec{n}$ be the normal vector of the desired plane.
Since plane must contain x-axis, $\vec{n}$ must be perpendicular to
direction vector $\vec{i}$ of x-axis, so $\vec{n} \cdot \vec{i} = 0$. The angle condition is

$\frac{\vec{n} \cdot \langle 1,1,0 \rangle}{\|\vec{n}\| \, \|\langle 1,1,0 \rangle\|} = \cos\frac{\pi}{3}$. Since we take $\|\vec{n}\| = 1$, we have the equations:

$\vec{n} = \langle a, b, c \rangle$

$$\begin{cases} \langle a,b,c \rangle \cdot \vec{i} = 0 \\ \dfrac{\langle a,b,c \rangle \cdot \langle 1,1,0 \rangle}{1 \cdot \sqrt{1^2+1^2}} = \cos\dfrac{\pi}{3} \\ a^2+b^2+c^2 = 1 \end{cases} \qquad \begin{cases} a = 0 \\ \dfrac{a+b}{\sqrt{2}} = \dfrac{1}{2} \Rightarrow b = \dfrac{\sqrt{2}}{2} \\ a^2+b^2+c^2=1 \quad \left(\dfrac{\sqrt{2}}{2}\right)^2 + c^2 = 1 \end{cases} \quad c^2 = \frac{1}{2}, \text{ so } c = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

There are two solutions for $\vec{n}$: $\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$, $\langle 0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$ giving us the

planes $\frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z = 0$, $\frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}z = 0$

(both have to contain the x-axis, in particular, $(0,0,0)$ is a pt. in the plane)