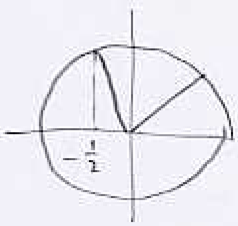
Spring '06/MAT 145/Exam 3 **Name:** Solution *Show all your work.*

1. (5pts) Use an addition formula to find the exact value of $\cos 165°$ (do not use the calculator).

$$\cos 165° = \cos(120° + 45°) = \cos 120° \cos 45° - \sin 120° \sin 45°$$

$$= -\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = -\frac{\sqrt{2}+\sqrt{6}}{4}$$

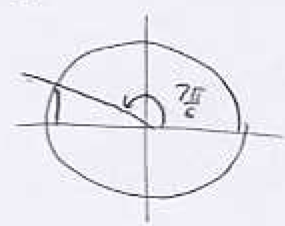


2. (3pts) Find the exact value of the expression (do not use the calculator):

$$\frac{\tan 57° + \tan 12°}{1 - \tan 57° \tan 12°} = \tan(57° - 12°) = \tan 45° = 1$$

3. (5pts) Use a half-angle formula to find the exact value of $\cos \frac{7\pi}{12}$ (do not use the calculator).

$$\frac{7\pi}{12} = \frac{1}{2}\cdot\frac{7\pi}{6} = \frac{\frac{7\pi}{6}}{2} \qquad \cos^2 \frac{\frac{7\pi}{6}}{2} = \frac{1+\cos\frac{7\pi}{6}}{2}$$



$$= \frac{1 - \frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2} = \frac{2-\sqrt{3}}{4}$$

So: $\cos^2 \frac{7\pi}{12} = \frac{2-\sqrt{3}}{4}$

$$\cos\frac{7\pi}{12} = \pm\sqrt{\frac{2-\sqrt{3}}{4}} = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\cos\frac{7\pi}{12} = -\frac{\sqrt{2-\sqrt{3}}}{2}$$ since $\frac{7\pi}{12}$ is in quad 2

4. (9pts) Suppose that $-\frac{\pi}{2} < \alpha < 0$ and $\frac{\pi}{2} < \beta < \pi$ are angles so that $\tan\alpha = -\frac{5}{3}$ and $\sin\beta = \frac{2}{7}$. Use addition and half-angle formulas to find:

a) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= -\frac{5}{\sqrt{34}}\left(-\frac{\sqrt{45}}{7}\right) - \frac{3}{\sqrt{34}}\cdot\frac{2}{7}$$

$$= \frac{5\sqrt{45} - 6}{7\sqrt{34}}$$

$$= \frac{15\sqrt{5} - 6}{7\sqrt{34}}$$

b) $\sin\frac{\alpha}{2}$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$$= \frac{1 - \frac{3}{\sqrt{34}}}{2}\cdot\frac{\sqrt{34}}{\sqrt{34}}$$

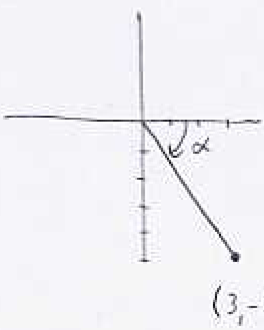
$$= \frac{\sqrt{34}-3}{2\sqrt{34}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{\sqrt{34}-3}{2\sqrt{34}}}$$

$$\sin\frac{\alpha}{2} = -\sqrt{\frac{\sqrt{34}-3}{2\sqrt{34}}}$$

since $-\frac{\pi}{4} < \frac{\alpha}{2} < 0$

$\tan\alpha = -\frac{5}{3} = \frac{y}{x} = \frac{-5}{3}$



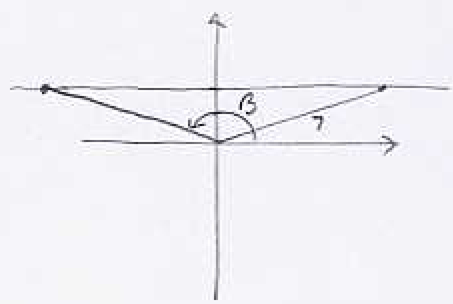
$r^2 = x^2 + y^2$

$r^2 = 3^2 + (-5)^2$

$r^2 = 34$

$r = \sqrt{34}$

$\sin\beta = \frac{2}{7} = \frac{y}{r}$



$x^2 + 2^2 = 7^2$

$x^2 + 4 = 49$

$x^2 = 45$

$x = \pm\sqrt{45}$

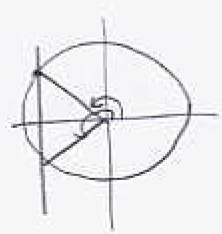
$x = -\sqrt{45}$

5. (4pts) Solve the equation (give a general formula for all the solutions).

$2\cos\theta + \sqrt{3} = 0$

$2\cos\theta = -\sqrt{3}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

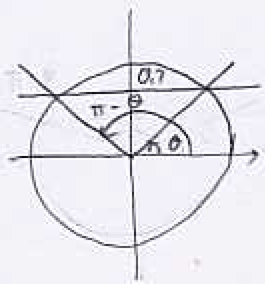


$\theta = \frac{5\pi}{6} + k\cdot 2\pi$

$\theta = \frac{7\pi}{6} + k\cdot 2\pi$

6. (5pts) Use your calculator to solve the equation on the interval $0 \leq \theta < 2\pi$. Round answers to two decimal places (answers in radians).

$\sin\theta = 0.7$



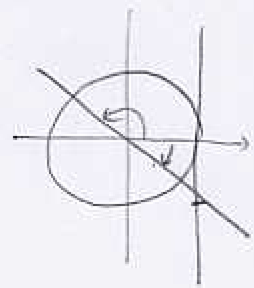
$\theta = \arcsin 0.7 \approx 0.78$

or $\theta = \pi - \arcsin 0.7 \approx 2.37$

7. (6pts) Solve the equation on the interval $-\pi \leq \theta \leq \pi$:

$\sin\theta + \cos\theta = 0$

$\sin\theta = -\cos\theta \quad | \div \cos\theta$



$\frac{\sin\theta}{\cos\theta} = -1$

$\tan\theta = -1$

$\theta = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$

8. (8pts) Solve the equation (give a general formula for all the solutions).

$\sin\theta + 1 = 2\cos^2\theta$

$\sin\theta + 1 = 2(1 - \sin^2\theta)$

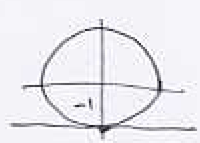
$\sin\theta + 1 = 2 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$2x^2 + x - 1 = 0$

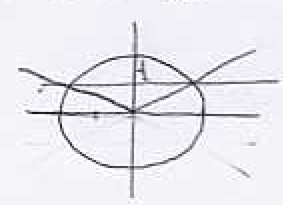
$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-1 \pm \sqrt{9}}{4} = -1, \frac{1}{2}$

$\sin\theta = -1$



$\theta = -\frac{\pi}{2} + k \cdot 2\pi$

$\sin\theta = \frac{1}{2}$



$\theta = \frac{\pi}{6} + k \cdot 2\pi$

$\theta = \frac{5\pi}{6} + k \cdot 2\pi$

9. (5pts) Establish the identity: $\dfrac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} = \cos\theta$

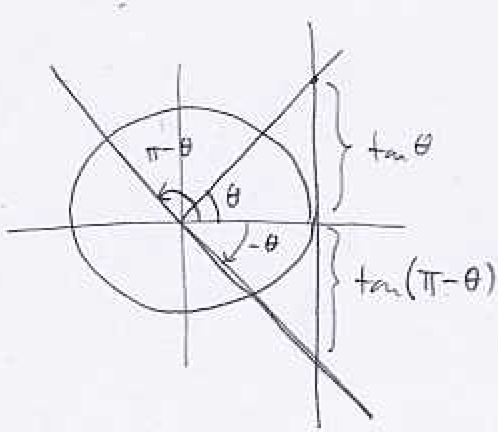
$$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\dfrac{1-\cos\theta}{1+\cos\theta}}{1+\dfrac{1-\cos\theta}{1+\cos\theta}} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \frac{1+\cos\theta-(1-\cos\theta)}{1+\cos\theta+(1-\cos\theta)}$$

$$= \frac{2\cos\theta}{2} = \cos\theta$$

Bonus. (5pts) Establish the identity $\tan(\pi-\theta) = -\tan\theta$ in two ways:
a) by using an addition formula
b) by drawing a picture and explaining.

a) $\tan(\pi-\theta) = \dfrac{\overset{0}{\overset{\shortparallel}{\tan\pi}}-\tan\theta}{1+\underset{\underset{0}{\shortparallel}}{\tan\pi}\tan\theta} = -\dfrac{\tan\theta}{1} = -\tan\theta$

b)



By symmetry these two lengths are equal, but the lower one counts as negative, so

$\tan(\pi-\theta) = -\tan\theta$