Fall '06/MAT 220/Worksheet 5 **Name:** Solution *Show all your work.*

1. (11pts) Let $f(x) = \dfrac{2}{x^2+1}$.

a) Find the x-intercepts and the y-intercept.
b) Find the domain and vertical and horizontal asymptotes, if any.
c) Find the intervals of increase and decrease and find the local extrema.
d) Find the intervals where the function is concave up/down and find the inflection points.
e) Sketch a nice graph of the function that takes into account everything you found in a)-d).

a) y-int: $f(0) = \dfrac{2}{0+1} = 2$

x-int. $\dfrac{2}{x^2+1} = 0$ no sol.

b) $x^2+1=0$

$x^2=-1$ no sol. $D = \mathbb{R}$

no vertical asymptotes

Horizontal: $f(x)$ behaves like $\frac{2}{x^2}$ for large $|x|$,

$\lim\limits_{x\to\infty} \dfrac{2}{x^2} = 0$, so

$y=0$ is a horizontal asymptote

c) $f'(x) = \dfrac{d}{dx}\left(2(x^2+1)^{-1}\right) = -2(x^2+1)^{-2}\cdot 2x$

$= -\dfrac{4x}{(x^2+1)^2}$ ← sign depends only on x; $(x^2+1)^2$ ← positive

$4x=0$
$x=0$

		0	
f'	+	0	−
f	↗	loc. max	↘

$f(0)=2$

c) $f''(x) = -\dfrac{4(x^2+1)^2 - 4x\cdot 2(x^2+1)2x}{(x^2+1)^4}$

$= -\dfrac{4(x^2+1)(x^2+1-4x^2)}{(x^2+1)^4}$

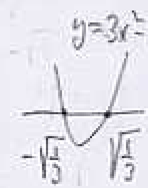
$= -\dfrac{4(-3x^2+1)}{(x^2+1)^3} = \dfrac{4(3x^2-1)}{(x^2+1)^3}$ ← sign depends only on this

$3x^2-1=0$

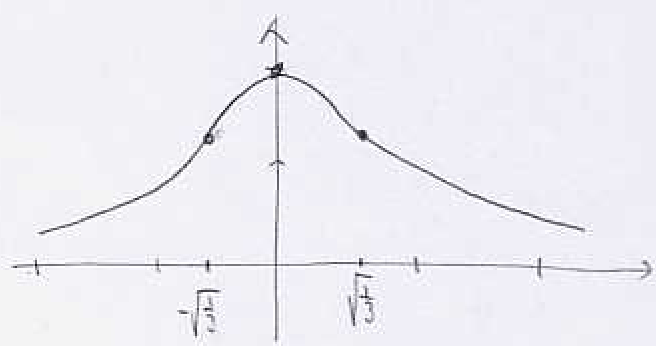
$3x^2=1$ $x = \pm\sqrt{\frac{1}{3}} \approx \pm 0.577$

$x^2 = \frac{1}{3}$

		$-\sqrt{\frac{1}{3}}$		$\sqrt{\frac{1}{3}}$	
$f''(x)$	+		−		+
$f(x)$	CU		CD		CU



e) $f\left(\pm\sqrt{\frac{1}{3}}\right) = \dfrac{2}{\left(\sqrt{\frac{1}{3}}\right)^2+1} = \dfrac{2}{\frac{1}{3}+1} = 2\cdot\frac{3}{4} = \frac{3}{2}$



2. (9pts) Let $f(x) = 3x^4 + 4x^3 - 72x^2 + 16$. Find the absolute extremes of this function (and where they occur) on the closed intervals below. Then draw on paper a rough graph of the function to verify your answers.

a) $[-2, 4]$

b) $[5, 7]$.

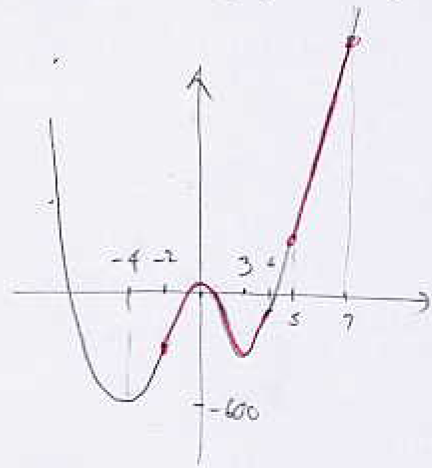
$f'(x) = 12x^3 + 12x^2 - 144x$

$12x^3 + 12x^2 - 144x = 0$

$12x(x^2 + x - 12) = 0 \quad | \div 12$

$x(x+4)(x-3) = 0$

$x = 0, -4, 3$



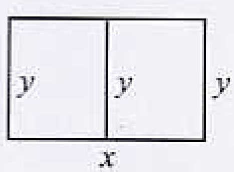
a)

x	$f(x)$	
-2	-256	
4	-112	
0	16	abs. max
3	-281	abs. min

b)

x	$f(x)$	
5	591	abs. min
7	5063	abs. max

3. (10pts) A farmer will use a fence to enclose a rectangular area and divide it in two parts. If the rectangle is to have area 500m^2, what should be its dimensions in order to minimize the length of the fence used? What is the minimal length of the fence? (When solving this problem, state the domain of your variable and use the second derivative test to ensure that your solution is indeed a minimum.)



$A = xy = 500\,m^2$

Fence used $2x + 3y$

$= 2x + 3 \cdot \frac{500}{x}$

Minimize $f(x) = 2x + \frac{1500}{x}$

on $0 < x$

$f'(x) = 2 - \frac{1500}{x^2}$

$2 - \frac{1500}{x^2} = 0$

$2x^2 = 1500$

$x^2 = 750$

$x = \pm\sqrt{750}$

$x = \sqrt{750} \approx 27.386$

$f''(x) = \frac{d}{dx}(2 - 1500x^{-2}) = 3000x^{-3}$

$= \frac{3000}{x^3}$

Clearly $f''(\sqrt{750}) > 0$ so f has a local min. at $x = \sqrt{750}$

Since there is just one critical point, it is an absolute minimum.

$x = \sqrt{750}\,m$, $y = \frac{500}{\sqrt{750}} = 18.257\,m$

Length of fence is $109.545\,m$