Note Title 9/29/2010

We compare equations 4 and 5

Runge-Kutta

$$x(t+h) = x(t) + (w_1 + w_2)hf + w_2(\alpha f_t + \beta f f_x)h^2 + O(h^3) \quad \cdots (4)$$

Taylor series

$$x(t+h) = x(t) + hf + \frac{1}{2}h^2(f_t + f f_x) + O(h^3) \quad (5)$$

For equation (4) and 5 to agree

$$\begin{cases} w_1 + w_2 = 1 \\ \alpha w_2 = \frac{1}{2} \\ \beta w_2 = \frac{1}{2} \end{cases} \quad (6)$$

A convenient solution is to take $\alpha = 1$ $\beta = 1$

$$w_1 = \frac{1}{2}, \quad w_2 = \frac{1}{2}$$

This yields a second order Runge-Kutta method

$$\begin{aligned} x(t+h) &= x(t) + w_1 h f(t,x) + w_2 h f(t+\alpha h, x+\beta h f(t,x)) \\ &= x(t) + \frac{h}{2} f(t,x) + \frac{1}{2} h f(t+h, x+hf(t,x)) \end{aligned}$$

that is

$$x(t+h) = x(t) + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = hf(t,x)$$

$$k_2 = hf(t+h, x+k_1)$$

Advantage: just two function evaluations of f.

Error term:

$$\frac{h^3}{4}\left(\frac{2}{3}-\alpha\right)\left(\frac{\partial}{\partial t}+f\frac{\partial}{\partial x}\right)^2 f + \frac{h^3}{6} f_x\left(\frac{\partial}{\partial t}+f\frac{\partial}{\partial x}\right) f$$

Example

Consider the IVP

$$\begin{cases} \dfrac{dx}{dt} = -tx^2 \\ x(0) = 2 \end{cases}$$

Use a second order Runge-Kutta method to compute $x(0.2)$ with stepsize $h = 0.1$

$$x(t+h) = x(t) + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = h f(t, x)$$

$$k_2 = h f(t+h, x+k_1)$$

Here $f(t, x) = -tx^2$

Now

$$x(0+0.1) = x(0.1) = \frac{1}{2}[k_1 + k_2]$$

$$k_1 = h f(0, x(0))$$

$$= 0.1 f(0, 2)$$

$$= 0.1 * (-0 * 2^2) = 0$$

$$k_2 = h f(0+h,\ x(0)+k_1)$$
$$= 0.1 * f(0.1,\ 2)$$
$$= 0.1 * (-0.1 * 2^2)$$
$$= -0.04$$

So $\quad x(0.1) = x(0) + \frac{1}{2}[k_1 + k_2]$

$$= 2 + \frac{1}{2}[0 - 0.04]$$
$$= 1.98$$

Next:

$$x(0.2) = x(0.1) + \frac{1}{2}(k_1 + k_2)$$

where $\quad k_1 = h f(0.1,\ x(0.1))$

$$= hf(0.1, 1.98)$$

$$= 0.1 * (-0.1 * 1.98^2)$$

$$= -0.0039204$$

$$k_2 = hf(t+h, x+k_1)$$

$$= 0.1 * f(0.2, 1.98 + (-0.0039204))$$

$$= 0.1 * f(0.2, 1.97607)$$

$$= 0.1 * (-0.2 * 1.97607^2)$$

$$=$$

Hence

$$x(0.2) = x(0.1) + \frac{1}{2}(k_1 + k_2)$$

Runge-Kutta Method of order 4

The classical fourth-order Runge-Kutta method is

$$x(t+h) = x(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(t, x)$$

$$k_2 = hf(t + \tfrac{1}{2}h, \; x + \tfrac{1}{2}k_1)$$

$$k_3 = hf(t + \tfrac{1}{2}h, \; x + \tfrac{1}{2}k_2)$$

$$k_4 = hf(t + h, \; x + k_3)$$

In computing $x(t+h)$, the main cost is evaluating the function f four times.

The final formula agrees with the Taylor expansion upto an including the term in h^4. [the error therefore contains h^5 but no lower powers of h].