Note Title 9/22/2010

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10 ORDINARY DIFFERENTIAL EQUATIONS

An ordinary differential equation (ODE) is an equation that involves one or more derivatives of an unknown function.

A solution of a differential equation is a specific function that satisfies the equation.

Example

ODE: $x' - x = e^t$

t - independent variable
x - dependent variable.

Solution: $x(t) = te^t + ce^t$ c- arbitrary constant.

[$x(t) = te^t + ce^t$

$\Rightarrow x'(t) = 1.e^t + \underbrace{te^t + ce^t}$

$= e^t + x(t)$

So

$x'(t) - x(t) = e^t$ (DE.)]

Example

$x'' + 9x = 0$ (DE)

$x(t) = C_1 \sin 3t + C_2 \cos 3t$ (Sol) C_1 & C_2 are Constants.

$$\Big[\, x'(t) = 3C_1 \cos 3t - 3C_2 \sin 3t$$

$$x''(t) = -9C_1 \sin 3t - 9C_2 \cos 3t$$

$$x'' + 9x = \left(-9C_1 \sin 3t - 9C_2 \cos 3t\right) + 9\left(C_1 \sin 3t + C_2 \cos 3t\right)$$

$$= 0 \,\Big]$$

A D.E does not, in general, determine a unique solution (appearance of constants).

We shall consider the initial value problem (IVP) for first order differential equation.

The standard form is

$$\begin{cases} x' = f(t,x) & \text{(DE)} \\ x(a) = b & \text{initial condition} \end{cases}$$

Since x is a function of t the DE is

$$\frac{d\,x(t)}{dt} = f(t, x(t)).$$

<u>Examples</u>

$$\begin{cases} x' = t \\ x(0) = 1 \end{cases} \qquad \boxed{f(t,x) = t} \qquad x = \frac{t^2}{2} + C$$

$$x' = t + x^2 \qquad \boxed{f(t,x) = t + x^2}$$

$$x(a) = b$$

Examples

① $\begin{cases} x' = x + 1 \\ x(0) = 0 \end{cases}$

Solution: $x = e^t - 1$

$x(0) = e^0 - 1 = 0$ ✓

$x' = e^t$

$x + 1 = (e^t - 1) + 1 = e^t$

IVP

② $\begin{cases} x' = 6t - 1 \\ x(1) = 6 \end{cases}$

Solution: $x = 3t^2 - t + 4$

$$x(1) = 3(1^2) - 1 + 4$$
$$= 3 - 1 + 4$$
$$= 6$$ ✓

Vector Fields

Consider the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(a) = b \end{cases}$$

The function f provides the slope of the solution in the tx-plane

At every point where $f(t,x)$ is defined, we can draw a short line segment through that point, having the prescribed slope.

We can draw as many as we wish to a get a diagram that illustrates discretely the vector field of the differential equation.

<u>Theorem 1</u> Uniqueness of IVP

If f and $\frac{\partial f}{\partial x}$ are continuous in the rectangle defined by $|t - t_0| < \alpha$ and $|x - x_0| < \beta$, then the initial value problem

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique continuous solution in some interval $|t - t_0| < \varepsilon$.

<u>Example</u>

Given the IVP

$$\begin{cases} x' = 1 + t\sin(tx) & 0 \le t \le 2 \\ x(0) = 0 \end{cases}$$

$$f(t,x) = 1 + t\sin(tx)$$

$$\frac{\partial f}{\partial x} = f_x = 0 + t\cos(tx) \cdot t$$

$$= t^2\cos(tx)$$

Both f and f_x are continuous for $0 \le t \le 2$ and for all x.

Hence the IVP has a unique continuous solution.