Note Title 9/20/2010

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Note that the eigenvalues of A and A^T are the same. [same characteristic polynomial].

<u>Corollary 3</u> More Gershgorin Discs

All eigenvalues of an $n \times n$ matrix $A = (a_{ij})$ are contained in the union of the discs $D_i = D_i(a_{ii}, s_i)$ in the complex plane having center a_{ii} and radii s_i given by the sum of the magnitudes of the columns of A.

i.e. $$\bigcup_{i=1}^{n} D_i = \bigcup_{i=1}^{n} \{ z \in \mathbb{C} : |z - a_{ii}| \leq s_i \}$$

where $$s_i = \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|$$

From the previous example with $A = \begin{bmatrix} -4 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 0 & 1 & 4 \end{bmatrix}$

$$D_1 = \{ z \in \mathbb{C} : |z - (-4)| \leq 2 \}$$
$$D_2 = \{ z \in \mathbb{C} : |z - 2| \leq 2 \}$$
$$D_3 = \{ z \in \mathbb{C} : |z - 3| \leq 2 \}$$
$$D_4 = \{ z \in \mathbb{C} : |z - 4| \leq 1 \}$$

The region containing the eigenvalues of A is

$$\left(\bigcup_{i=1}^{n} C_i\right) \bigcap \left(\bigcup_{i=1}^{n} D_i\right).$$

Corollary 4

For a matrix A, the union of k Gershgorin discs that do not intersect the remaining $n-k$ circles contain exactly k (counting multiplicities) of the eigenvalues of A.

Corollary 5

Every strictly diagonally dominant matrix is non singular.

[Zero cannot lie in any of its Gershgorin discs, so it must be invertible].

Singular Value Decomposition

Recall: The singular values of a matrix A are the non negative square roots of the eigenvalues of A^TA.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \boxed{\det(A^TA - \lambda I) = 0}$$

The eigenvalues of A^TA are $\lambda = 3$ and $\lambda = 1$.

The singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1}$

Definition

A singular value decomposition of an $m \times n$ matrix A is any representation of A in the form

$$A = UDV^T$$

where

U and V are orthogonal matrices

D is an $m \times n$ diagonal matrix having non negative diagonal entries that are ordered $\underset{\sigma_{11}}{d_{11}} \geq \underset{\sigma_{22}}{d_{22}} \geq \underset{\sigma_{33}}{d_{33}} \geq \dots \geq 0$

The diagonal entries d_{ii} are just the singular values of A.

we write
$$A = UDV^T = \underset{m \times m}{U} \underset{m \times n}{\begin{bmatrix} \sigma_{11} & & & & 0 \\ & \sigma_{22} & & & \\ & & \ddots & & \\ & & & \sigma_{nn} & \\ 0 & & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}} \underset{n \times n}{V^T}$$

We can express the condition number of a matrix in terms of its singular values

$$\kappa(A) = \sqrt{\frac{\sigma_{max}}{\sigma_{min}}}$$

[Recall

$$\kappa(A) = \|A\|_2 \, \|A^{-1}\|_2$$

$$\|A\|_2^2 = \rho(A^T A) = \sigma_{max} \qquad \|A\|_2 = \sqrt{\rho(A^T A)}$$

$$\|A^{-1}\|_2^2 = \rho((A^{-1})^T A^{-1}) = \frac{1}{\sigma_{min}}$$