September 17, 2010

Note Title 9/17/2010

Properties of Eigenvalues

Theorem 1

The following statements are true for any square matrix.

1 If λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue of $p(A)$, for any polynomial p. In particular, λ^k is an eigenvalue of A^k.

[If λ is an eigenvalue of A the $Ax = \lambda x$ for some non zero vector x. Want to show that $A^k x = \lambda^k x$.

Use induction: True for $k = 1$ i.e. $Ax = \lambda x$

Assume true for $n \geq 1$ i.e. $A^n x = \lambda^n x$

Now $A^{n+1} x = A A^n x = A(\lambda^n x)$ by assumption

$= \lambda^n A x = \lambda^n \lambda x = \lambda^{n+1} x$ True for all k.

2 If A is non singular and λ is an eigenvalue of A, then $p(\frac{1}{\lambda})$ is an eigenvalue of $p(A^{-1})$, for any polynomial p. In particular, λ^{-1} is an eigenvalue of A^{-1}.

λ is an eigenvalue of A i.e. $Ax = \lambda x$ for some $x \neq 0$.

$A^{-1}(Ax) = A^{-1}(\lambda x) \Rightarrow x = \lambda A^{-1} x \Rightarrow \frac{1}{\lambda} x = A^{-1} x$

3 If A is real and symmetric, then its eigenvalues are real.

4 If A is complex and Hermitian, then its eigenvalues are positive.

5 If A is Hermitian and positive definite, then its eigenvalues are positive.

Definition

Two matrices A and B are similar to be similar if there exists a non singular matrix P such that

$$B = PAP^{-1}.$$

Theorem 2

Similar matrices have the same eigenvalues.

Proof

Characteristic polynomial of B, $P_B(x)$

$$
\begin{aligned}
P_B(x) &= \det(B - xI) \\
&= \det(PAP^{-1} - xI) \quad \text{because } A \,\&\, B \text{ are similar.} \\
&= \det(P(A - xI)P^{-1})
\end{aligned}
$$

$$
\begin{aligned}
PxIP^{-1} &= x\,PIP^{-1} \\
&= x\,PP^{-1} \\
&= xI
\end{aligned}
$$

x is a scalar.

$$
\begin{aligned}
&= \det(P)\det(A - xI)\det(P^{-1}) \\
&= \det(P).\det(P^{-1})\det(A - xI) \quad \det(UV) = \det U.\det V \\
&= \det(A - xI) \\
&= \text{characteristic polynomial of } A.
\end{aligned}
$$

So similar matrices have the same characteristic polynomial.

Hence result follows (they have the same zeros! eigenvalues!)

Definition

A matrix U is unitary if $UU^* = I$.

$[U^* = \bar{U}^T]$

Matrices A and B are unitarily similar to each other if

$$B = U^* A U$$

for some unitary matrix U.

Theorem (Schur's Theorem)

Every square matrix is unitarily similar to a triangular matrix.

[Given an arbitrary matrix A, there exists a unitary matrix U such that

$$UAU^* = T$$

where

$UU^* = I$ and T is a triangular matrix.]

We may wish to determine where the eigenvalues of a matrix are situated in the complex plane $\mathbb{C}$.

Theorem 4: Gershgorin's Theorem.

All eigenvalues of an $n \times n$ matrix $A = (a_{ij})$ are contained in the union of the n discs

$$C_i = C_i(a_{ii}, r_i)$$

in the complex plane with center a_{ii} and radii r_i given by the sum of the magnitudes of the off-diagonal entries in the ith row.

The matrix A can have real or complex entries. The region containing the eigenvalues of A can be written as

$$\bigcup_{i=1}^{n} C_i = \bigcup_{i=1}^{n} \{ z \in \mathbb{C} : |z - a_{ii}| \le r_i \}$$

where

$$r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$$

Example

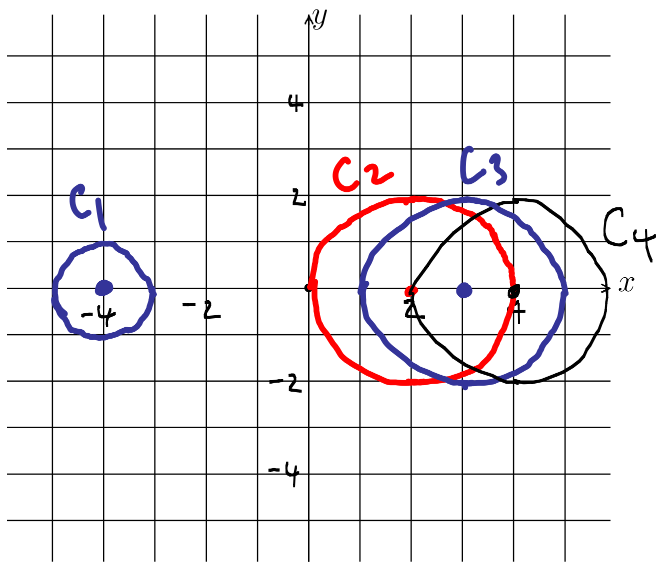
Let
$$A = \begin{bmatrix} -4 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 0 & 1 & 4 \end{bmatrix}$$

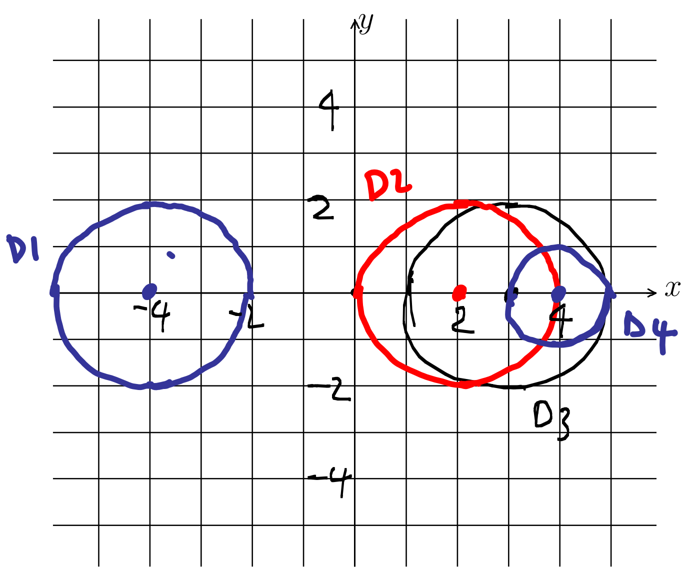
Discs are :

$$C_1 = \{ z \in \mathbb{C} : |z - (-4)| \le 1 \}$$ radius $r_1 = |0| + |0| + |1|$ Center $a_{11} = -4$

$$C_2 = \{ z \in \mathbb{C} : |z - 2| \le 2 \}$$
$$C_3 = \{ z \in \mathbb{C} : |z - 3| \le 2 \}$$
$$C_4 = \{ z \in \mathbb{C} \cdot |z - 4| \le 2 \}$$

$C_1 = \{z \in \mathbb{C} : |z - (-4)|$



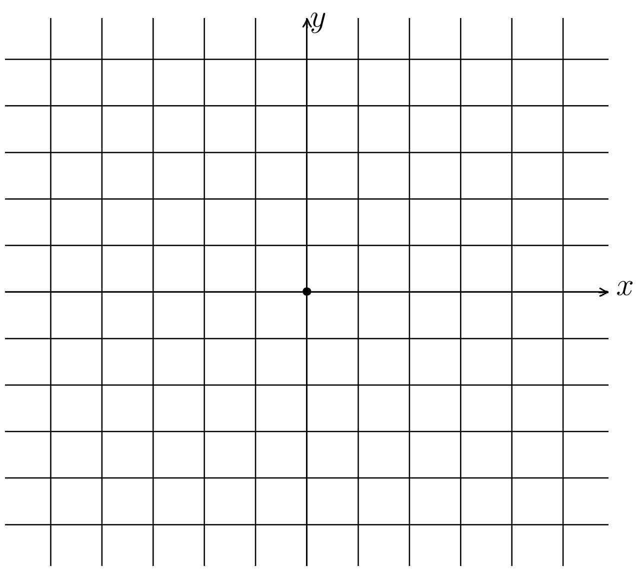


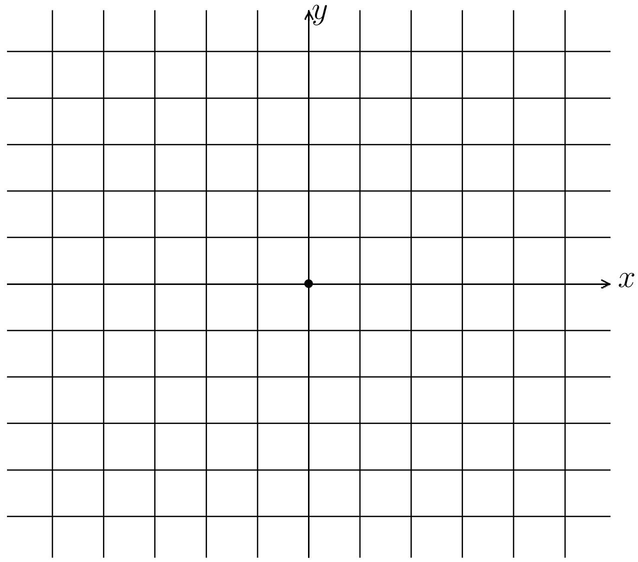
$D_1 = \{z \in \mathbb{C} : |z - (-4)| \leq 2\}$

$D_2 = \{z \in \mathbb{C} : |z - 2| \leq 2\}$

$D_3 = \{z \in \mathbb{C} : |z - 3| \leq 2\}$

$D_4 = \{z \in \mathbb{C} : |z - 4| \leq 1\}$

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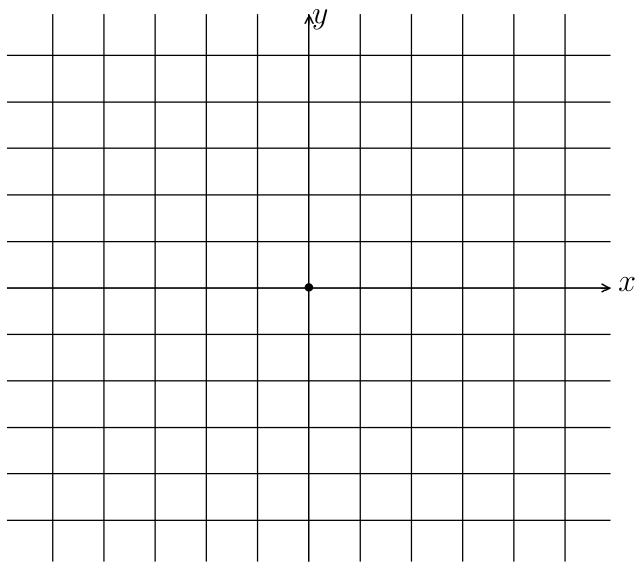
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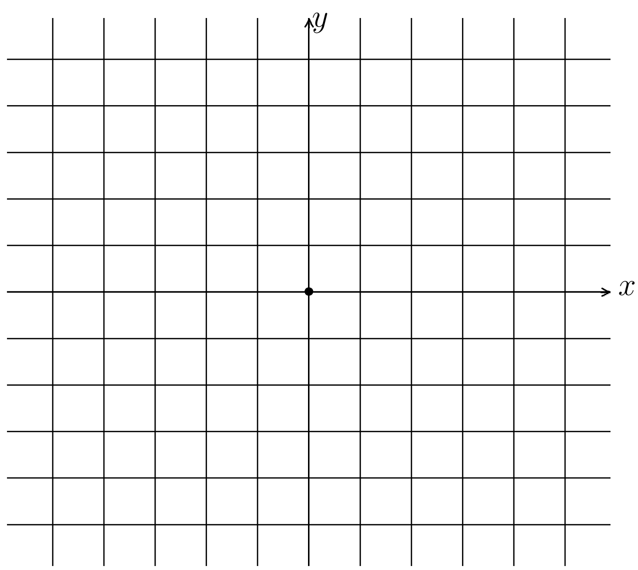
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