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Note Title 9/15/2010

## 8.3 Eigenvalues and Eigenvectors

**Question:**

Let $A$ be an $n \times n$ matrix. Are there nonzero vectors $v$ for which $Av$ is a scalar multiple of $v$?

i.e.

$$Av = \lambda v \ ?$$

scalar #.

## Example

Let $A = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$

Now

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\underset{v}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \underset{\lambda}{\underset{\uparrow}{5}}\underset{v}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

and

$$A\underset{v}{\begin{bmatrix} 2 \\ -7 \end{bmatrix}} = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -7 \end{bmatrix} = \begin{bmatrix} -8 \\ 28 \end{bmatrix} = \underset{\lambda}{-4}\underset{v}{\begin{bmatrix} 2 \\ -7 \end{bmatrix}}$$

## Definition

An $n \times n$ matrix $A$ is said to have an eigenvalue $\lambda \in \mathbb{C}$ if there is a non zero vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x.$$

The vector $x$ is called an eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

NOTE:

If $Ax = \lambda x$, $x \neq 0$, then every non zero multiple of $x$ is an eigenvector.

# Calculating Eigenvalues and eigenvectors

$$Ax = \lambda x, \quad x \neq 0.$$

$$\Rightarrow \quad Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

So $A - \lambda I$ is singular (non invertible)

Hence

$$\det(A - \lambda I) = 0$$ way to compute $\lambda$!

Define

$$p(\lambda) = \det(A - \lambda I)$$

Called the characteristic polynomial of $A$.

[ the eigenvalues of $A$ are just the zero of the characteristic polynomial ].

Once the eigenvalue is determined, an eigenvector can be determined by solving the system

$$(A - \lambda I)x = 0$$

## Example

Let $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

The characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{bmatrix} 0-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix}$$

$$= -\lambda(3-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

The eigenvalues are obtained from

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-2)(\lambda-1) = 0$$

$$\lambda = 2 \text{ or } \lambda = 1$$

The eigenvalues are $\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 2}$.

Eigenvectors ?

For $\lambda_1 = 1$

Solve

$$(A - \lambda_1 I)x = 0$$

$$(A - I)x = 0$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x_1 - x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = -x_2 \\ x_1 = -x_2 \end{array}$$

Let $x_1 = 1$ then $x_2 = -1$.

A corresponding eigenvector $x = [x_1, x_2]^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\underbrace{\text{For } \lambda_2 = 2}$ :

$$(A - \lambda_2 I)x = 0$$

$$(A - 2I)x = 0$$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{cases} -2x_1 - x_2 = 0 \\ 2x_1 + x_2 = 0 \end{cases} \Rightarrow -2x_1 = x_2$$

let $x_1 = 1$. then $x_2 = -2$

Correspoind eigenvector is $x = [1 \ -2]^T$

<u>Example</u>

let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

$$p(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \left| \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} \right| + 2 \left| \begin{bmatrix} 0 & 1-\lambda \\ -1 & 1 \end{bmatrix} \right|$$

$$= (1-\lambda) \left( (1-\lambda)^2 + 1 \right) + 2(1-\lambda)$$

$$= (1-\lambda) \left[ (1-\lambda)^2 + 1 + 2 \right]$$

$$= (1-\lambda) \left[ (1-\lambda)^2 + 3 \right]$$

Eigenvalues:

$$p(\lambda) = 0$$

$$(1-\lambda)\left[(1-\lambda)^2+3\right]=0$$

either $\lambda = 1$ or $(1-\lambda)^2+3 = 0$

$$1-\lambda = \pm\sqrt{-3}$$

$$1-\lambda = \pm\sqrt{3}\,i$$

$\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = 1+\sqrt{3}\,i}$ $\boxed{\lambda_3 = 1-\sqrt{3}\,i}$

Eigenvectors?

For $\lambda_1 = 1$ $(A-\lambda_1 I)x = 0$

$$(A-I)x = 0$$

So

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x_3 = 0 \\ -x_3 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Rightarrow \quad x_3 = 0 \quad x_2 = x_1$$

let $x_1 = 1$ then $x_2 = 1$

The associated eigenvector is $x = [1 \ 1 \ 0]^T$

Since $\lambda_2$ and $\lambda_3$ are complex numbers, their corresponding eigenvectors are also complex.

# Properties of Eigenvalues

Recall:

A is symmetric if $A^T = A$.

For complex matrix A define the conjugate transpose $A^* = \bar{A}^T = (\bar{a}_{ji})$.

$$A = \begin{bmatrix} 1-i & 3 \\ 2 & 2+3i \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} \overline{1-i} & \bar{3} \\ \bar{2} & \overline{2+3i} \end{bmatrix}$$

$$= \begin{bmatrix} 1+i & 3 \\ 2 & 2-3i \end{bmatrix}$$

Positive definite $x^T A x > 0$ for all non zero vectors $x$.

<u>Theorem 1</u>

The following statements are true for any square matrix $A$.

1. If $\lambda$ is an eigenvalue of $A$, then $p(\lambda)$ is an eigen value of $p(A)$, for any polynomial $p$. In particular, $\lambda^k$ is an eigenvalue of $A^k$.