Note Title 9/1/2010

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Theorem 2. Jacobi and Gauss-Seidel Convergence theorem.

If A is diagonally dominant, then the Jacobi and Gauss-Seidel methods converge for any starting vector $x^{(0)}$. [Sufficient but not a necessary condition].

Definition Symmetric Positive Definite

Matrix A is symmetric positive definite if

$$A^T = A \quad \text{(Symmetric)}$$

and

pos. def $x^T A x > 0$ for all non zero real vectors x.

Theorem

A matrix A is symmetric positive definite iff $A^T = A$ and all eigenvalues of A are positive.

Proof

($\Rightarrow$)

Assume that A is symmetric positive definite.

$$\begin{cases} x^T A x > 0 & \text{for all } x \in \mathbb{R}^n \ \ x \neq 0. \\ A^T = A \end{cases}$$

Need to show that all eigenvalues of A are positive.

Let λ be an eigenvalue of A.

$Ax = \lambda x$ for some non zero vector x.

Have

$$0 < x^T A x = x^T \lambda x$$

because λ is a scalar.

$$= \lambda x^T x$$

$$= \lambda \underbrace{\|x\|_2^2}_{\text{positive number.}}$$

$0 < \lambda \|x\|_2^2$ so λ is positive.

$(\Leftarrow)$

assume $A^T = A$ and all eigenvalues are positive

Need to show that A is symmetric positive definite

i.e.

$$x^T A x > 0 \quad \text{for all non zero } x \in \mathbb{R}^n.$$

Let λ be an eigen value.

$$Ax = \lambda x$$

$$x^T A x = x^T \lambda x$$

$$x^T A x = \lambda x^T x$$

$$= \lambda \underbrace{\|x\|_2^2}_{\text{positive \#}} > 0$$

positive

$\boxed{//}$

Theorem 3 SOR Convergence Theorem

Suppose that the matrix A has positive diagonal elements and that $0 < \omega < 2$. The SOR method converges for any starting vector $x^{(0)}$ if and only if A is symmetric and positive definite.

Example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Clearly $A^T = A$

Check eigenvalues!! positive?

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b$$

Matrix Formulation

we can split the matrix A as

$$A = D - L - U$$

where D - non zero diagonal matrix

L - a strictly lower triangular matrix

U - a strictly upper triangular.

Example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ then } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; L = -\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \; U = -\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Our linear system $Ax = b$ then may be expressed as $(D - L - U)x = b$

The Jacobi method in matrix form is

$$Dx^{(k)} = (L + U)x^{(k-1)} + b$$

i.e. $Q = D$

The Gauss-Seidel method becomes

$$(D - L)x^{(k)} = Ux^{(k-1)} + b \quad \text{i.e. } Q = D - L$$

The SOR method can be written as

$$(D - \omega L)x^{(k)} = [\omega U + (1-\omega)D]\,x^{(k-1)} + \omega b$$

Recap! Recall the error formula

$$e^{(5)} = (\qquad)\,e^{(4)}$$

$$e^{(4)} = (\qquad)\,e^{(3)}$$

$$\begin{aligned}
e^{(k)} &= (\underline{I} - Q^{-1}A)\,e^{(k-1)} \\
&= (\underline{I} - Q^{-1}A)(\underline{I} - Q^{-1}A)\,e^{(k-2)} \\
&= (\underline{I} - Q^{-1}A)^2\,e^{(k-2)} \\
&= (\underline{I} - Q^{-1}A)^2(\underline{I} - Q^{-1}A)\,e^{(k-3)} \\
&= (\underline{I} - Q^{-1}A)^3\,e^{(k-3)}
\end{aligned}$$

$$\vdots$$

$$= (I - Q^{-1}A)^{k-1} e^{(1)}$$

$$e^{(k)} = (I - Q^{-1}A)^{k} e^{(0)}$$ the k^{th} iteration error in terms of the initial error.

Take norms

$$\|e^{(k)}\| = \|(I - Q^{-1}A)^{k} e^{(0)}\|$$

$$\leq \|I - Q^{-1}A\|^{k} \|e^{(0)}\|$$

If $\|I - Q^{-1}A\| < 1$ then $\|I - Q^{-1}A\|^{k} \xrightarrow[\text{as } k \text{ increases}]{} 0$

So $\|e^{(k)}\| \longrightarrow 0$ i.e. get smaller and smaller.

Convergence!!