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$y = ax + b$

Note Title 10/27/2010

Nonpolynomial Example

The least squares method is not restricted to linear (first-degree) polynomials or to any specific functional form.

Suppose we want to fit a table of values (x_k, y_k) where $k = 0, 1, 2, \ldots, m$, by a function of the form

$$y = a \ln x + b \cos x + c e^x$$

in the least squares sense.

• we need to determine the coefficients a, b, and c.

So we consider the function

$$\varphi(a,b,c) = \sum_{k=0}^{m} (a \ln x_k + b \cos x_k + c e^{x_k} - y_k)^2 .$$

At minimum

$$\frac{\partial \varphi}{\partial a} = \sum_{k=0}^{m} 2(a \ln x_k + b \cos x_k + c e^{x_k} - y_k) \ln x_k = 0$$

$$\frac{\partial \varphi}{\partial b} = \sum_{k=0}^{m} 2(a \ln x_k + b \cos x_k + c e^{x_k} - y_k) \cos x_k = 0$$

$$\frac{\partial \varphi}{\partial c} = \sum_{k=0}^{m} 2(a \ln x_k + b \cos x_k + c e^{x_k} - y_k) e^{x_k} = 0$$

We get the three normal equations

$$\begin{cases} a\sum_{k=0}^{m}(\ln x)^2 + b\sum_{k=0}^{m}(\cos x_k)(\ln x_k) + c\sum_{k=0}^{m} e^{x_k}\ln x_k = \sum_{k=0}^{m} y_k \ln x_k \\ a\sum_{k=0}^{m}(\ln x_k)(\cos x_k) + b\sum_{k=0}^{m}(\cos x_k)^2 + c\sum_{k=0}^{m}(e^{x_k})(\cos x_k) = \sum_{k=0}^{m} y_k \cos x_k \\ a\sum_{k=0}^{m}(\ln x_k)e^{x_k} + b\sum_{k=0}^{m}(\cos x_k)e^{x_k} + c\sum_{k=0}^{m}(e^{x_k})^2 = \sum_{k=0}^{m} y_k e^{x_k} \end{cases}$$

Example

Fit a function of the form

$$y = a\ln x + b\cos x + c e^x$$

to the following table values.

x	0.24	0.65	0.95	1.24	1.73	2.01	2.23
y	0.23	-0.26	-1.10	-0.45	0.27	0.10	-0.29

2.52	2.77	2.99
0.24	0.56	1.00

L1	L2	L3	L4	L5	L6
x_k	y_k	$(\ln x_k)^2$	$(\ln x_k)(\cos x_k)$	$(\ln x_k)e^{x_k}$	$(\cos x_k)^2$
0.24	0.23				
0.65	-0.26				
0.95	-1.10				
1.24	-0.45				
1.73	0.27				
2.01	0.10				
2.23	-0.29				
2.52	0.24				
2.77	0.56				
2.99	1.00				

$\Sigma x_k = 17.33$ $\Sigma y_k = 0.3$ $\sum_{k=0}^{9} (\ln x_k)^2 = 6.79410$

$\Sigma (\ln x_k)(\cos x_k) = -5.34749$ $\Sigma (\ln x_k) e^{x_k} = 63.258$ (89.)

$(\cos x_k)\, e^{x_k}$ $\quad (e^{x_k})^2$ $\quad y_k \ln x_k$ $\quad y_k \cos x_k$ $\quad y_k e^{x_k}$

$$\sum_{k=0}^{9} (\cos x_k)^2 = 5.10842$$

$$\sum (e^{x_k})^2 = 1002.50650 \qquad \sum y_k \ln x_k = 1.161627$$

$$\sum y_k \cos x_k = -2.38271 \qquad \sum \cos x_k e^{x_k} = -49.60859$$

$$\sum y_k e^{x_k} = 26.77277$$

Hence

$$\begin{cases} 6.794102a - 5.347496b + 63.25889c = 1.61624 \\ -5.347496a + 5.108425b - 49.00859c = -2.38271 \\ 63.25889a - 49.00859b + 1002.50650c = 26.77277 \end{cases}$$

Solve the system to get

$$a = -1.04103, \quad b = -1.26132, \quad c = 0.03073$$

$$y = -1.04103 \ln x - 1.26132 \cos x + 0.03073 e^x$$