Note Title 10/20/2010

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11.2 Higher-Order Equations

Consider the initial-value problem for ordinary differential equations of order higher than 1.

$$\begin{cases} x^{(n)} = f(t, x, x', x'', \ldots, x^{(n-1)}) \\ x(a),\ x'(a),\ x''(a), \ldots, x^{(n-1)}(a) \text{ all given} \end{cases}$$

A differential equation of order n is normally accompanied by n auxiliary conditions. This many initial conditions are needed to specify the solution of the d.e. precisely.

<u>Examples</u>

$$\begin{cases} x'' - 2x' + 2x = e^{2t}\sin t & 0 \le t \le 1 \\ x(0) = -0.4 \\ x'(0) = -0.6 \end{cases}$$

This is a second order ode

$$f(t, x, x') = e^{2t}\sin t - 2x + 2x'$$

Example

$$\begin{cases} x'' - 2x' + x = te^t - t & 0 \le t \le 1 \\ x(0) = x'(0) = 0 \end{cases}$$

actual solution:

$$x(t) = \frac{1}{6}t^3e^t - te^t + 2e^t - t - 2$$

Example

$$\begin{cases} x''' + 2x'' - x - 2x = e^t & 0 \le t \le 3 \\ x(0) = 1 \\ x'(0) = 2 \\ x''(0) = 0 \end{cases}$$

actual solution

$$x(t) = \frac{43}{36}e^t + \frac{1}{4}e^{-t} - \frac{4}{9}e^{-2t} + \frac{1}{6}te^t$$

A higher order ode with initial conditions is solved numerically by turning it into a system of first-order differential equations.

Consider

$$\begin{cases} x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)}) \\ x(a),\ x'(a),\ x''(a) \dots,\ x^{(n-1)}(a) \quad \text{all give} \end{cases}$$

Define new variables $x_1, x_2, x_3, \dots, x_n$ as follows:

$$x_1 = x$$

$$x_2 = x'$$

$$x_3 = x''$$

$$\vdots$$

$$x_{n-1} = x^{(n-2)}$$

$$x_n = x^{(n-1)}$$

Note that

$$x_1' = x' = x_2$$

$$x_2' = x'' = x_3$$

$$\vdots$$

$$x_{n-1}' = x^{(n-1)} = x_n$$

$$x_n' = x^{(n)} = f(t, x, x', x'', \cdots, x^{(n-1)})$$

$$= f(t, x_1, x_2, x_3, \cdots, x_n)$$

So the original initial value problem may then be written as a system of first-order d.e.

$$\left\{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \dots, x_n) \\ &x_1(a),\ x_2(a),\ x_3(a),\ \dots,\ x_n(a) \quad \text{all given} \end{aligned}\right.$$

Example

Transform the IVP

$$\begin{cases} x''' = \cos x + \sin x' - e^{x''} + t^2 \\ x(0) = 3, \quad x'(0) = 7 \quad x''(0) = 13 \end{cases}$$

into a form suitable for solution by the Runge-Kutta procedure.

Let $x_1 = x$

$x_2 = x'$

$x_3 = x''$

So we have the system of first order odes.

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = \cos x_1 + \sin x_2 - e^{x_3} + t^2 \\ x_1(0) = 3 \quad x_2(0) = 7 \quad x_3(0) = 13 \end{cases}$$

<u>Example</u>

$$\begin{cases} x'' - 2x' + x = e^{2t} \sin t \qquad 0 \le t \le 1 \\ x(0) = -0.4 \\ x'(0) = -0.6 \end{cases}$$

The equivalent system of first-order odes is:

Let $x_1 = x$

$x_2 = x'$

then

$$\begin{cases} x_1' = x_2 \\ x_2' = e^{2t}\sin t - x_1 + 2x_2 \\ x_1(0) = -0.4 \quad x_2(0) = -0.6 \end{cases}$$

Example

$$\begin{cases} x''' + 2x'' - x' - 2x = e^t \qquad 0 \le t \le 3 \\ x(0) = 1 \\ x'(0) = 2 \\ x''(0) = 0 \end{cases}$$

Let $x_1 = x$

$x_2 = x'$

$x_3 = x''$

then

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = e^t + 2x_1 + x_2 - 2x_3 \\ x_1(0) = 1, \quad x_2(0) = 2, \quad x_3(0) = 0 \end{cases}$$