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Note Title 10/15/2010

11 Systems of Ordinary Differential Equations

11.1 Methods for First-Order Systems.

$$\begin{cases} x'(t) = f(t, x) \\ x(a) = b \end{cases}$$

We consider a system of n first-order differential equations with n conditions which we can write in the form

$$\begin{cases} x_1'(t) = f_1(t, x_1, x_2, \dots, x_n) \\ x_2'(t) = f_2(t, x_1, x_2, \dots, x_n) \\ \quad\vdots \\ x_n'(t) = f_n(t, x_1, x_2, \dots, x_n) \\ x_1(a) = s_1, \quad x_2(a) = s_2, \quad \dots, \quad x_n(a) = s_n \end{cases}$$

For example, consider the system of two equations

$$\begin{cases} x' = y \\ y' = x \end{cases} \quad \text{with initial conditions} \quad \begin{cases} x(0) = -1 \\ y(0) = 0 \end{cases}$$

It is not possible to solve either of the two differential equations by itself because the first equation governing x' involves the unknown function y, and the second equation governing y' involves the unknown function x.

The two equations are said to be <u>coupled</u>.

The analytic solution is

$$x(t) = -\cosh t$$

$$y(t) = -\sinh t$$

Check: $x'(t) = \frac{d}{dt}(-\cosh t)$

$\cosh t = \frac{e^{t} + e^{-t}}{2}$

$= -\sinh t$

$= y(t)$

$\sinh t = \frac{e^{t} - e^{-t}}{2}$

$y'(t) = \frac{d}{dt}(-\sinh t)$

$= -\cosh t$

$= x(t)$

$x(0) = -\cosh(0) = -1$

$y(0) = -\sinh(0) = 0$

Taylor Series Method

Let us extend the Taylor series method to the system

$$\begin{cases} x' = y \\ y' = x \end{cases} \qquad (2)$$

$$x(0) = -1, \quad y(0) = 0$$

Recall that

$$x(t+h) = x(t) + hx' + \frac{h^2}{2}x'' + \frac{h^3}{6}x''' + \cdots$$

$$y(t+h) = y(t) + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \cdots$$

For the second order Taylor series method

$$\begin{cases} x(t+h) = x + hx' + \frac{h^2}{2}x'' \\ y(t+h) = y + hy' + \frac{h^2}{2}y'' \end{cases}$$

But

$$x' = y \qquad \text{and} \qquad y' = x$$

$$x'' = y' = x \qquad\qquad y'' = x' = y$$

Hence

$$x(t+h) = x + hy + \frac{h^2}{2}x$$

$$y(t+h) = y + hx + \frac{h^2}{2}y$$

If we take $h = 0.1$ then

$$x(0.1) = x(0) + h\,y(0) + \frac{h^2}{2}x(0)$$
$$= -1 + (0.1)(0) + \frac{(0.1)^2}{2}(-1)$$
$$= \boxed{-1.005}$$

$$y(0.1) = y(0) + h\,x(0) + \frac{h^2}{2}y(0)$$
$$= 0 + (0.1)(-1) + 0$$
$$= -0.1$$

$$\begin{cases} x(0.2) = x(0.1) + h\,y(0.1) + \frac{h^2}{2}x(0.1) \\ y(0.2) = y(0.1) + h\,x(0.1) + \frac{h^2}{2}y(0.1) \end{cases}$$

<u>Remark</u>

Note that the system of ODEs can be written in vector notation as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the system can be written as

$$\begin{cases} X' = F(t, X) \\ X(0) = S \end{cases}$$

where

$$S = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \& \quad F(t, X) = \begin{bmatrix} y \\ x \end{bmatrix}.$$

For the general system of n first-order differential equations if we let:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad X' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \quad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad S = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

then the system can be written in vector form as

$$\begin{cases} X' = F(t, X) \\ X(a) = S \end{cases}$$

Runge-Kutta Methods

The Runge-Kutta methods extend to systems of d.e.

The classical fourth-order Runge-Kutta for the system

$$\begin{cases} X' = F(t, X) \\ X(a) = S \end{cases}$$

is

$$X(t+h) = X + \frac{h}{6}(K_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = F(t, X)$$
$$k_2 = F(t + \tfrac{1}{2}h, X + \tfrac{1}{2}hk_1)$$

$$k_3 = F\left(t + \tfrac{1}{2}h,\ X + \tfrac{1}{2}hk_2\right)$$

$$k_4 = F\left(t + h,\ X + hk_3\right)$$

Except for the variables t and h, all quantities are vectors with n components.

For example, consider the IVP

$$X = F(t, X)$$

$$\begin{cases} x' = y \\ y' = x \\ x(0) = -1 \\ y(0) = 0 \end{cases} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

Here $F(t, X) = \begin{bmatrix} y \\ x \end{bmatrix}$ $\qquad F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$

X

Take $h = 0.1$, Then

$$K_1 = F(0, X(0)) = \begin{bmatrix} y(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$K_2 = F\left(t + \tfrac{1}{2}h,\ X + \tfrac{1}{2}h k_1\right)$$

$$= F\left(t + \tfrac{1}{2}h,\ X(0) + \tfrac{1}{2}(0.1)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$= F\left(t + \tfrac{1}{2}h,\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0.05\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$= F\left(t + \frac{1}{2}h, \begin{bmatrix} -0.95 \\ -0.05 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -0.05 \\ -0.95 \end{bmatrix}$$