Note Title 11/29/2010

15.3 Elliptic Problems

$x u_{xx} + y u_{xy} + u_{yy} = 0$

$A = x \quad B = y \quad C = 1$

$B^2 - 4AC > 0$ hyperbolic

$B^2 - 4AC = 0$ parabolic

$B^2 - 4AC < 0$ elliptic

$\boxed{y^2 - 4x(1)}$

hyperbolic if $y^2 - 4x > 0$ $\quad y^2 > 4x$

parabolic if $y^2 - 4x = 0$ $\quad y^2 = 4x$ $\quad x \geq 0$

elliptic if $y^2 - 4x < 0$ $\quad y^2 < 4x$ $\quad x > 0$

15.3 Elliptic Problems

Examples

Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

Poisson equation

$$\nabla^2 u = g(x, y)$$

Helmholtz Equation model Problem

Assume that on a given region R in the xy-plane

$$\begin{cases} \nabla^2 u + f u = g \\ u(x,y) \text{ known on the boundary of } R. \end{cases}$$

$f = f(x,y)$ and $g(x,y)$ are given continuous functions defined in R.

When f is a constant, the partial differential equation is called the Helmholtz equation.

Finite Difference Method

Recall finite difference approximation

$$f''(x) \approx \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right]$$

Assume constant spatial steplength h for both x and y

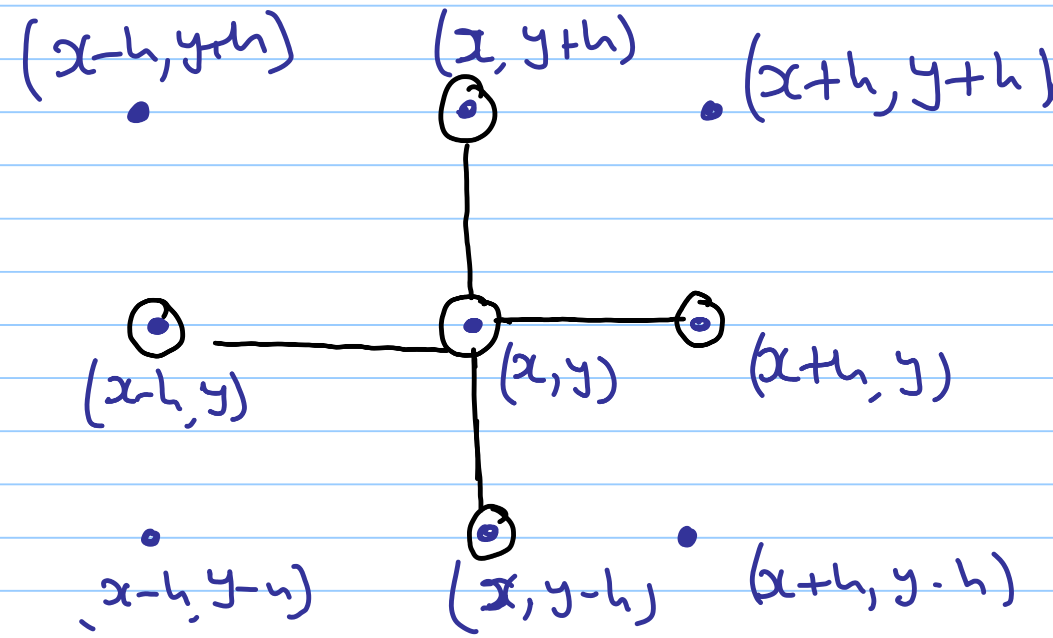
So

$$\nabla^2 u(x,y) \approx \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right] + \frac{1}{h^2}\left[u(x,y+h) - 2u(x,y) + u(x,y-h)\right]$$

$$= \frac{1}{h^2}\left[u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)\right]$$

This gives the five point approximation

Stencil.



The local error is

$$-\frac{h^2}{12}\left[\frac{\partial^4 u(\xi,y)}{\partial x^4} + \frac{\partial^4 u(x,\eta)}{\partial y^4}\right]$$

The approximation is of order $O(h^2)$.

Recall the problem

$$\begin{cases} \nabla^2 u + f u = g \\ u(x,y) \text{ known on the boundary } \partial R. \end{cases}$$

We solve using finite differences

using constant spatial length for x and y

$$x_i = ih \qquad y_j = jh \qquad (i, j \geq 0)$$

and let

$$u_{ij} = u(x_i, y_j) \qquad f_{ij} = f(x_i, y_j) \qquad g_{ij} = g(x_i, y_j)$$

then

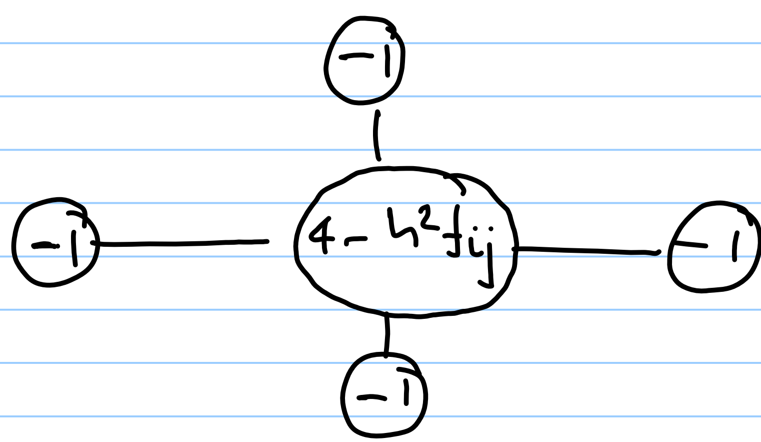
$$\frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right] + f_{ij}\,u_{ij} = g_{ij}$$

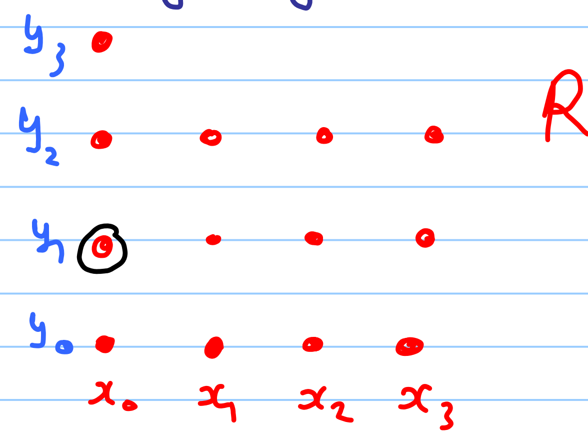
1-c.

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} + h^2 f_{ij} u_{ij} = h^2 g_{ij}$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - (4 - h^2 f_{ij}) u_{ij} = h^2 g_{ij}$$

$$-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + (4 - h^2 f_{ij}) u_{ij} = -h^2 g_{ij}$$





$u_{0,1} = u(x_0, y_1)$ boundary

$i=1, j=1$:

$$-u_{21} - u_{01} - u_{1,2} - u_{1,0} + (4 - h^2 f_{11}) u_{11} = -h^2 g_{11}$$

$$-u_{21} - u_{12} + (4 - h^2 f_{11}) u_{11} = -h^2 g_{11} + u_{01} + u_{10}$$

$$-u_{12} + (4 - h^2 f_{11}) u_{11} - u_{21} = -h^2 g_{11} + u_{01} + u_{10}$$

$i=1, j=2$:

$$-u_{22} - \underline{\underline{u_{0,2}}} - u_{1,3} - u_{1,1} + (4 - h^2 f_{12}) u_{1,2} = -h^2 g_{12}$$

boundary.