Note Title 11/22/2010

Numerical Solution – Hyperbolic Problems

We consider the model problem

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

we approximate the derivatives by finite differences choosing step sizes h and k for x and t respectively

$$u_{tt}(x,t) \approx \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2}$$

$$u_{xx}(x,t) \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

So we have

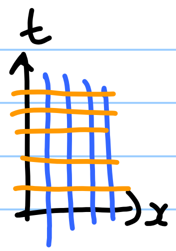
$$\frac{1}{k^2}\left[u(x,t+k) - 2u(x,t) + u(x,t-k)\right] = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

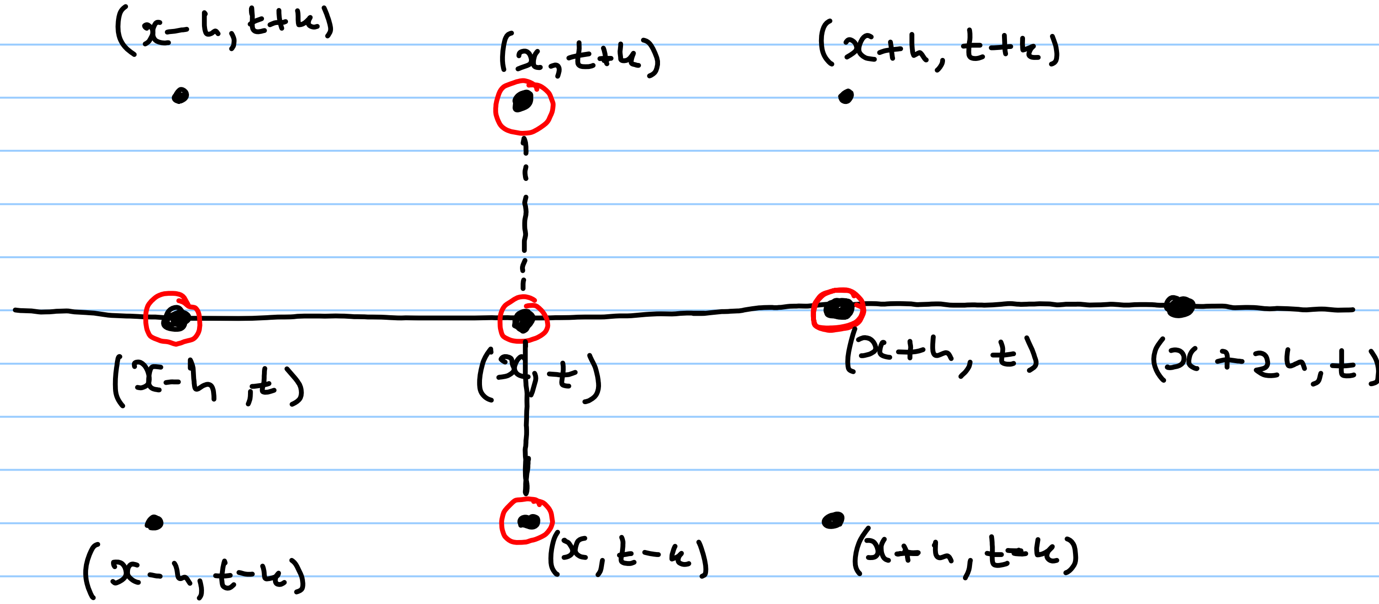
$$\Rightarrow u(x,t+k) = \frac{k^2}{h^2}u(x+h,t) + 2\left(1 - \frac{k^2}{h^2}\right)u(x,t)$$
$$+ \frac{k^2}{h^2}u(x-h,t) - u(x,t-k)$$

let $\rho = \frac{k^2}{h^2}$

$$u(x, t+k) = \rho\, u(x+h, t) + 2(1-\rho)\, u(x,t) + \rho\, u(x-h, t) - u(x, t-k)$$

Stencil:





Note that $u_t(x, 0) \simeq \dfrac{u(x, 0+k) - u(x, 0)}{k}$

So the boundary conditions can be written as

$$\begin{cases} u(x,0) = f(x) \\ \frac{1}{k}\left[u(x,k) - u(x,0)\right] = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

The finite difference problem is now solved starting at $t=0$ where u is known and then for $t=k$, $t=2k$, $t=3k$, ...

Note that from the initial condition

$$\frac{1}{k}\left[u(x,k) - u(x,0)\right] = 0$$

We have
$$u(x,k) = u(x,0) = f(x).$$
Approximating u_t by first differences, i.e. $O(k)$ leads to low accuracy in the computed solution.

Consider using central difference approximation

$$u_t(x,t) \simeq \frac{u(x,t+k) - u(x,t-k)}{2k}$$

So

$$u_t(x,0) \simeq \frac{u(x,k) - u(x,-k)}{2k}$$

The initial condition

$$u_t(x,0) = 0$$

becomes $\frac{1}{2k}\left[u(x,k) - u(x,-k)\right] = 0$

that is

$$u(x,k) = u(x,-k) \quad (*)$$

This requires a row of grid points $(x,-k)$. To solve this problem

Consider

$$u(x,t+k) = \rho u(x+h,t) + 2(1-\rho)u(x,t) + \rho u(x-h,t) - u(x,t-k)$$

let $t=0$

$$u(x,k) = \rho\, u(x+h,0) + 2(1-\rho)\, u(x,0) + \rho\, u(x-h,0) - u(x,-k)$$

$$= \rho f(x+h) + 2(1-\rho) f(x) + \rho f(x-h) - u(x,-k)$$

$$u(x,k) + \underbrace{u(x,-k)}_{u(x,k)} = \rho f(x+h) + 2(1-\rho) f(x) + \rho f(x-h)$$

from boundary condition

$$2u(x,k) = \rho\left[f(x+h) + f(x-h)\right] + 2(1-\rho) f(x)$$

$$\boxed{u(x,k) = \tfrac{1}{2}\rho\left[f(x+h) + f(x-h)\right] + (1-\rho) f(x)}$$

Subsequent values of $u(x,t)$ for $t = nk$ $(n \geq 2)$ can now be computed using formula for $u(x, t+k)$.