15.2 Hyperbolic Problems

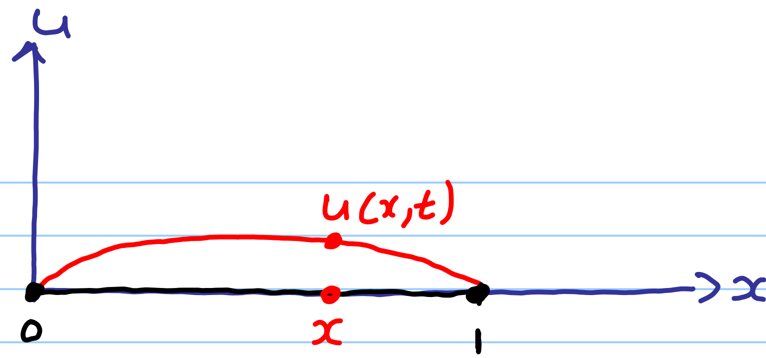
Note Title 11/19/2010

Wave Equation Model Problem

The wave equation with one space variable is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

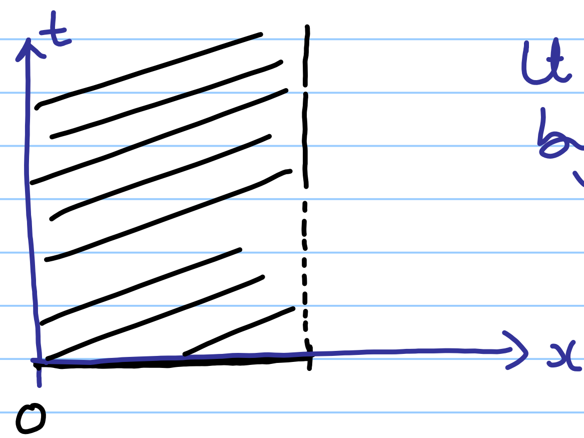
It governs the vibrations of a string. $u(x,t)$ represents the deflection at time t of a point on the string whose coordinate is x when the string is at rest.



Consider a model problem where the points on the string have coordinates x on the interval $0 \le x \le 1$, defined by the BVP

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = f(x) & \text{deflection at time } t = 0 \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 & \text{ends of string remain fixed.} \end{cases}$$

A solution is sought in the xt-plane



It is the semi-infinite strip defined by $0 \le x \le 1$ and $t \ge 0$

Analytic Solution

Assume that

- f posses two derivatives

- $f(-x) = -f(x)$ odd function
- $f(x+2) = f(x)$ periodic

$\Rightarrow f$ is extended over the whole real line.

Then the analytic solution to

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

is

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] \quad (*)$$

Let us verify that equation (*) is indeed the solution.

(1) $u_x = \frac{1}{2}\left[f'(x+t)\cdot(1) + f'(x-t)\cdot(1)\right]$

$$u_{xx} = \frac{1}{2}\left[f''(x+t)\cdot(1) + f''(x-t)\cdot(1)\right]$$

$$u_t = \frac{1}{2}\left[f'(x+t)\cdot(1) + f'(x-t)\cdot(-1)\right]$$

$$u_{tt} = \frac{1}{2}\left[f''(x+t)(1) - f''(x-t)\cdot(-1)\right]$$

$$= \frac{1}{2}\left[f''(x+t) + f''(x-t)\right]$$

Clearly $u_{tt} = u_{xx}$

2. $u(x,0) = \frac{1}{2}[f(x+0) + f(x-0)]$

$= \frac{1}{2}[f(x) + f(x)] = f(x)$

$u_t(x,0) = \frac{1}{2}[f'(x+0) - f'(x-0)]$

$= \frac{1}{2}[f'(0) - f'(0)]$

$= 0$

3. $u(0,t) = \frac{1}{2}[f(0+t) + f(0-t)]$

$= \frac{1}{2}[f(t) + f(-t)]$

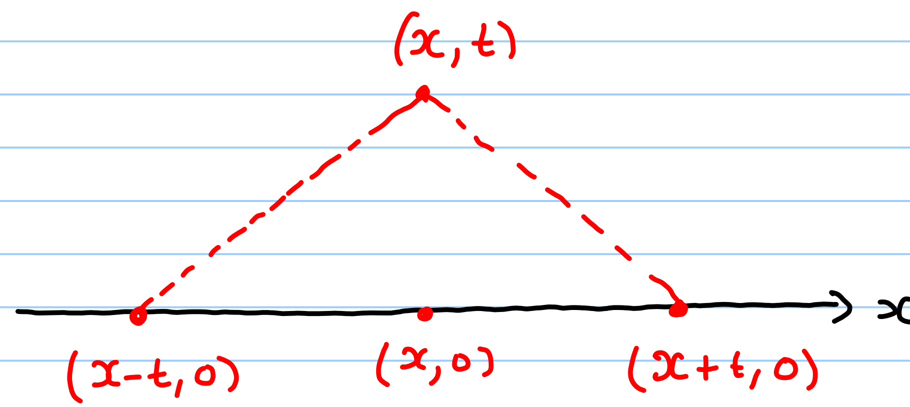
$= \frac{1}{2}[f(t) - f(t)]$ f is odd

$= 0$

$$
\begin{aligned}
u(1,t) &= \tfrac{1}{2}\left[f(1+t) + f(1-t)\right] \\
&= \tfrac{1}{2}\left[f(1+t) + f(-(t-1))\right] \\
&= \tfrac{1}{2}\left[f(1+t) - f(t-1)\right] && f \text{ is odd} \\
&= \tfrac{1}{2}\left[f(1+t) - f(t-1+2)\right] && f \text{ is periodic, period} = 2. \\
&= \tfrac{1}{2}\left[f(1+t) - f(t+1)\right] \\
&= 0
\end{aligned}
$$

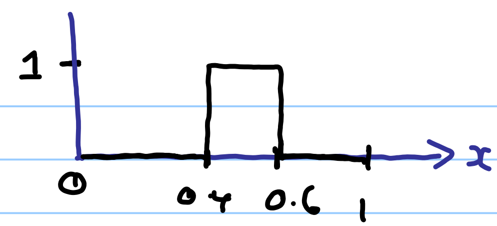
verification complete!

To compute $u(x,t)$ one only needs to know f at only two points on the x-axis, namely $x+t$ and $x-t$.

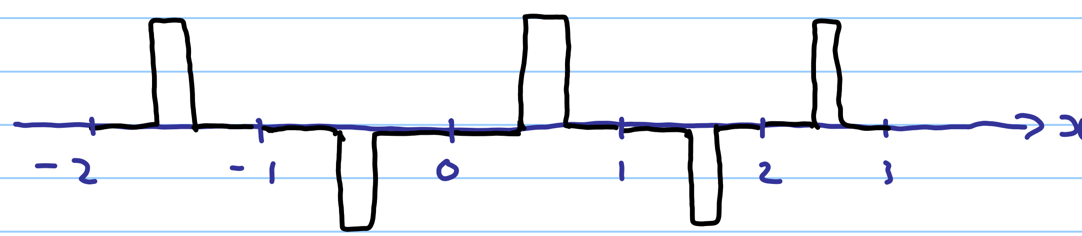


Example

Suppose $f(x) =$



Over the whole real line



Now let us find $u(x, 0.5)$

$$u(x,0.5) = \frac{1}{2}\left[f(x+0.5) + f(x-0.5)\right]$$

$$u(0,0.5) = \frac{1}{2}\left[f\left(\tfrac{1}{2}\right) + f\left(-\tfrac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 + (-1)\right] = 0$$

$$u(0.2,0.5) = \frac{1}{2}\left[f(0.7) + f(-0.3)\right] = \frac{1}{2}\left[0+0\right] = 0$$

$$u(0.1,0.5) = \frac{1}{2}\left[f(0.6) + f(-0.4)\right] = \frac{1}{2}\left[1 + (-1)\right] = 0$$