15 Partial Differential Equations

Examples from applied problems

- The wave equation.

  Let u represent the displacement at time t of a particle whose position at rest is (x, y, z)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$u = u(x, y, z, t)$$

- The heat equation, let u represent the temperature at time t in a physical body at the point that has coordinates (x, y, z)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

- Laplaces equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

the steady-state distribution of heat in a body.

- Navier-Stokes equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

Where u & v are components of the velocity vector in a fluid flow.

p – pressure

the fluid is assumed to be incompressible but viscous

Operator :

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{(grad)}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{(Laplacian operator)}$$

Δ

Example

Heat equation $\quad \frac{1}{\kappa} \frac{\partial u}{\partial t} = \nabla^2 u$

wave equation $\quad \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \nabla^2 u$

Laplace equation $\nabla^2 u = 0$

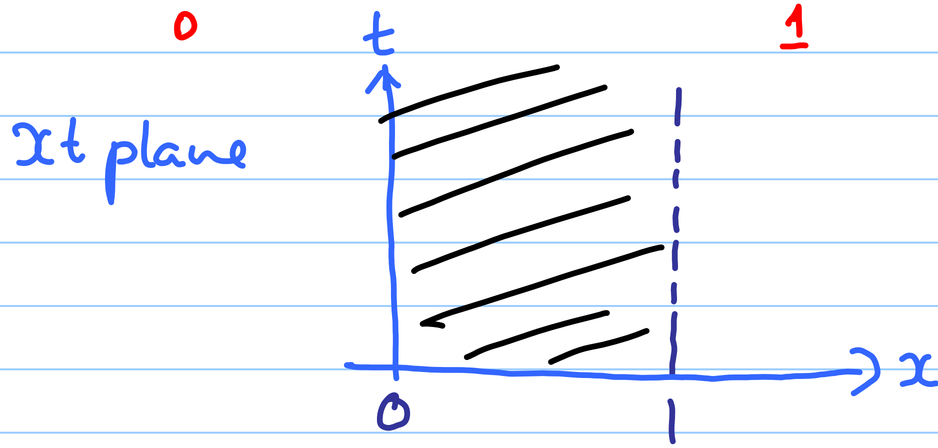
Poisson equation $\nabla^2 u = -4\pi\rho$

15.1 Parabolic Problems

We consider the heat equation in one spatial variable together with some boundary conditions

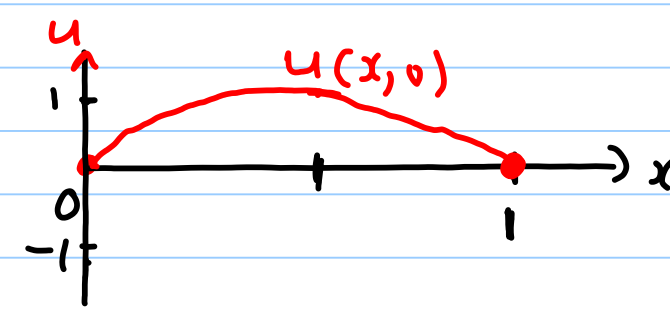
$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial}{\partial t} u(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sin(\pi x) \end{cases}$$

The equation describes the temperature $u(x,t)$ in a thin rod of length 1 when the ends are held at temperature 0.



$u(x,t)$

$u(x,0) = \sin \pi x$



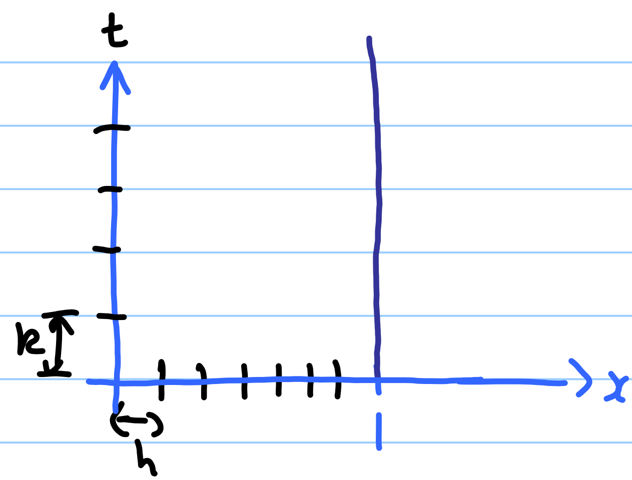
Finite Difference method

The derivatives are approximated by finite differences.

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x, t+k) - u(x,t)}{k}$$

where k is the time step

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x+h, t) - 2u(x,t) + u(x-h,t)}{h^2}$$



h is the spatial step length.

So

$$\frac{1}{h^2}\left[u(x+h,t) - 2u(x,t) + u(x-h,t)\right]$$

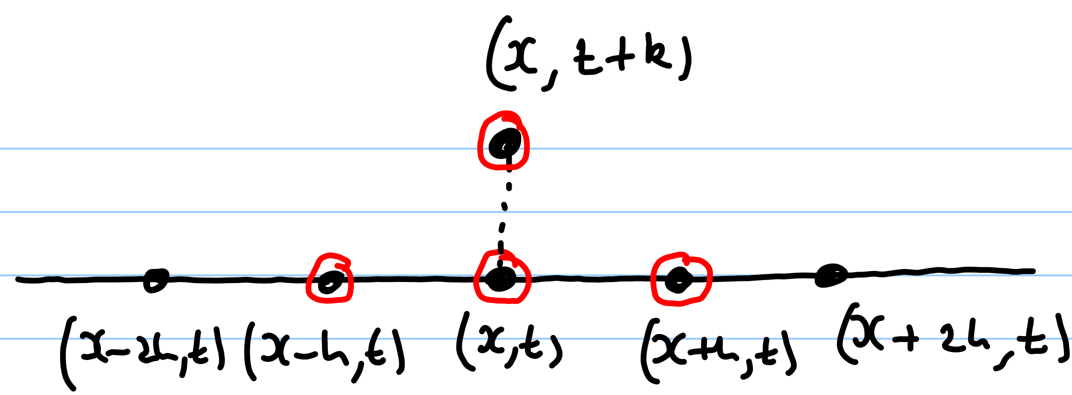
$$= \frac{1}{k}\left[u(x,t+k) - u(x,t)\right]$$

that is

$$u(x,t+k) = u(x,t) + \frac{k}{h^2}\left[u(x+h,t) - 2u(x,t) + u(x-h,t)\right]$$

$$= \sigma u(x+h,t) + (1-2\sigma)u(x,t) + \sigma u(x-h,t)$$

where $\sigma = \frac{k}{h^2}$



The solution marches forward in time

Analysis of the method shows that for stability

$$1 - 2\sigma \geq 0$$