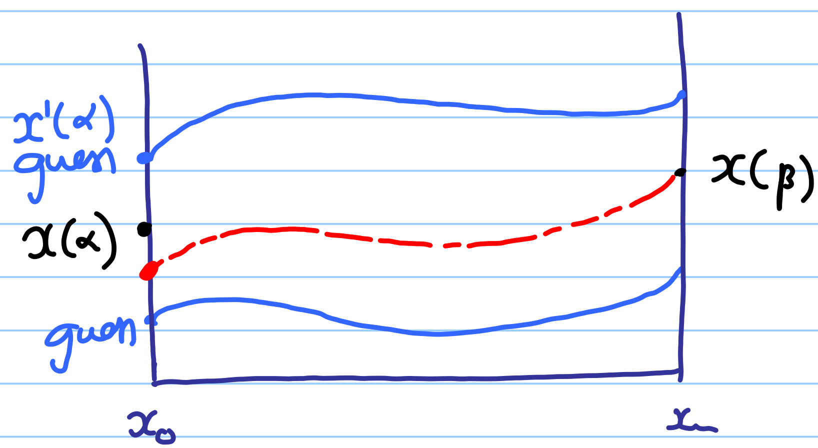
BVP

Shooting method.



14.2 A Discretization Method

Recall:

$$x(t+h) = x(t) + h x'(t) + \frac{h^2}{2} x''(t) + O(h^3)$$

$$x(t-h) = x(t) - h x'(t) + \frac{h^2}{2} x''(t) + O(h^3)$$

using first differences:

$$\begin{cases} x(t+h) \approx x(t) + h x'(t) \\ x(t-h) \approx x(t) - h x'(t) \end{cases}$$

Subtract second from first

$$x(t+h) - x(t-h) \approx 2h x'(t)$$

So can approximate 1st derivative by

$$\boxed{x'(t) \approx \frac{1}{2h}\left[x(t+h) - x(t-h)\right]}$$

$x-h$ $\quad x \quad$ $x+h$

Central difference

Also

$$\begin{cases} x(t+h) \approx x(t) + h x'(t) + \frac{h^2}{2} x''(t) \\ x(t-h) \approx x(t) - h x'(t) + \frac{h^2}{2} x''(t) \end{cases}$$

Adding

$$x(t+h) + x(t-h) \approx 2x(t) + h^2 x''(t)$$

Can approximate second derivative by central difference

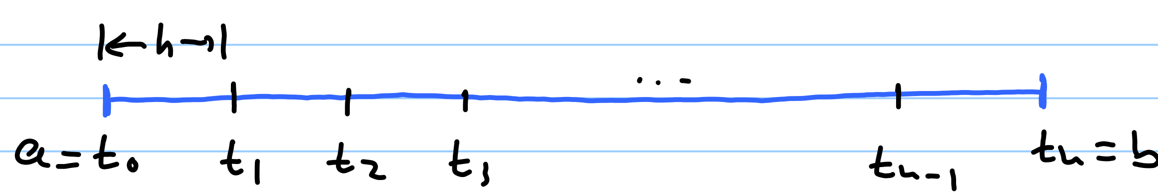
$$\boxed{x''(t) = \frac{1}{h^2}\left[x(t-h) - 2x(t) + x(t+h)\right]}$$

We want to solve

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha . \quad x(b) = \beta \end{cases}$$

- Select a set of equally spaced points $t_0, t_1, t_2, \ldots, t_n$ on the interval $[a,b]$ by letting

$$t_i = a + ih \quad \text{with} \quad h = \frac{b-a}{n} \qquad (0 \le i \le n)$$



- Next, approximate the derivatives by the standard central difference formulae

$$x'(t) = \frac{1}{2h}\left[x(t+h) - x(t-h)\right]$$

$$x''(t) = \frac{1}{h^2}\left[x(t-h) - 2x(t) + x(t+h)\right]$$

denote the approximate value of $x(t_i)$ by x_i.

So the differential equation is discretized as

$$x_0 = \alpha$$

when $i=1$: $\frac{1}{h^2}(x_2 - 2x_1 + x_0) = f\left(t_1, x_1, \frac{1}{2h}(x_2 - x_0)\right)$

$$x'' = f(t, x, x')$$

$$\frac{1}{h^2}\left[x(t+h) - 2x(t) + x(t-h)\right] = f\left(t, x(t), \frac{1}{2h}\left(x(t+h) - x(t-h)\right)\right)$$

$$\frac{1}{h^2}\left[x(t_1+h) - 2x(t_1) + x(t_1-h)\right] = f\left(t_1, x(t_1), \cdots\right)$$

$$\frac{1}{h^2}\left[\underset{x_2}{\underset{\approx}{x(t_2)}} - 2\underset{x_1}{\underset{\approx}{x(t_1)}} + \underset{x_0}{\underset{\approx}{x(t_0)}}\right] = f(\quad)$$

when $i = 2$:

$$\frac{1}{h^2}\left(x_3 - 2x_2 + x_1\right) = f\left(t_2, x_2, \frac{1}{2h}(x_3 - x_1)\right)$$

and the process continues to $i = n-1$.

The problem becomes

$$\begin{cases} x_0 = \alpha \\ \frac{1}{h^2}\left(x_{i-1} - 2x_i + x_{i+1}\right) = f\left(t_i, x_i, \frac{1}{2h}\left(x_{i+1} - x_{i-1}\right)\right) & 0 \le i \le n-1 \\ x_n = \beta \end{cases}$$

Which is usually a non linear system of equations with $n-1$ unknowns $x_1, x_2, x_3, \dots, x_{n-1}$ (f usually nonlinear in x_i)

When the right hand side is linear i.e.

$$f(t, x, x') = u(t) + v(t)\,x + w(t)\,x'$$

then

$$\frac{1}{h^2}\left[x_{i-1} - 2x_i + x_{i+1}\right] = u(t_i) + v(t_i)\,x_i + w(t_i)\left[\frac{1}{2h}\left(x_{i+1} - x_{i-1}\right)\right]$$

let

$$u_i = u(t_i), \quad v_i = v(t_i), \quad w_i = w(t_i).$$

then

$$\frac{1}{h^2}\left[x_{i-1} - 2x_i + x_{i+1}\right] = u_i + v_i x_i + w_i \frac{1}{2h}\left(x_{i+1} - x_{i-1}\right) \qquad 0 \le i \le n-1$$

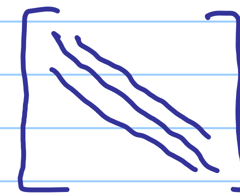
Collect like terms and multiply by $(-h^2)$

$$\underbrace{-h^2\left(\frac{1}{h^2} + w_i\frac{1}{2h}\right)}_{a_i} x_{i-1} \underbrace{-(h^2)\left(\frac{2}{h^2} + v_i\right)}_{b_i} x_i + \underbrace{(h^2)\left(\frac{1}{h^2} - w_i\frac{1}{2h}\right)}_{c_i} x_{i+1}$$

$$= -h^2 u_i \qquad 0 \le i \le n-1$$

i.e. $a_i x_{i-1} + b_i x_i + c_i x_{i+1} = u_i \qquad 0 \le i \le n-1$

We get a tridiagonal system.

 $\begin{bmatrix} x_{i-1} \\ x_i \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$