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Note Title 11/1/2010

<u>Example</u> Chebyshev Polynomials

The Chebyshev polynomials for the interval $[-1,1]$ are defined by

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_j(x) = 2x\,T_{j-1}(x) - T_{j-2}(x) \qquad (j \geq 2) \end{cases}$$

Alternatively $T_k(x) = \cos(k \arccos x)$

The first few Chebyshev polynomials are

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x\,T_1(x) - T_0(x)$$

$$= 2x \cdot x - 1$$

$$= 2x^2 - 1$$

$$T_3(x) = 2x\,T_2(x) - T_1(x)$$

$$= 2x\left[2x^2 - 1\right] - x$$

$$= 4x^3 - 2x - x$$

$$\Rightarrow \quad T_3(x) = 4x^3 - 3x$$

$$\begin{aligned} T_4(x) &= 2xT_3(x) - T_2(x) \\ &= 2x[4x^3 - 3x] - [2x^2 - 1] \\ &= 8x^4 - 8x^2 + 1 \end{aligned}$$

etc.

If we assume that the data points in the least-squares problem have the property

$$-1 = x_0 < x_1 < \cdots < x_m = 1$$

then the Chebyshev polynomials can be used.

Linear combinations of Chebyshev polynomials are easy to evaluate.

Consider an arbitrary linear combination of $T_0, T_1, T_2, \ldots, T_n$:

$$g(x) = \sum_{j=0}^{n} c_j T_j(x)$$

For any x, we compute $g(x)$ by

$$\begin{cases} w_{n+2} = w_{n+1} = 0 \\ w_j = c_j + 2x\,w_{j+1} - w_{j+2} \qquad j = n, n-1, \dots, 0 \\ g(x) = w_0 - x\,w_1 \end{cases}$$

Proof

$$g(x) = \sum_{j=0}^{n} c_j\, T_j(x)$$

$$= \sum_{j=0}^{n} \left(w_j - 2x\,w_{j+1} + w_{j+2} \right) T_j$$

$$= \sum_{j=0}^{n} w_j T_j - 2x \sum_{j=0}^{n} w_{j+1} T_j + \sum_{j=0}^{n} w_{j+2} T_j$$

Shifting indices

$$= \sum_{j=0}^{n} w_j T_j - 2x \sum_{j=1}^{n+1} w_j T_{j-1} + \sum_{j=2}^{n+2} w_j T_{j-2}$$

$$= w_0 T_0 + w_1 T_1 + \sum_{j=2}^{n} w_j T_j - 2x\, w_1 T_0 - 2x \sum_{j=2}^{n} w_j T_{j-1}$$

$$- 2x\, w_{n+1} T_n + \sum_{j=2}^{n} w_j T_{j-2} + w_{n+1} T_{n-1} + w_{n+2} T_n$$

$$= w_0 \underset{1}{\underset{\shortparallel}{T_0}} + w_1 \underset{x}{\underset{\shortparallel}{T_1}} - 2x\, w_1 \underset{1}{\underset{\shortparallel}{T_0}} - 2x \underbrace{w_{n+1}}_{=0} T_n + \underbrace{w_{n+1}}_{=0} T_{n-1} + \underbrace{w_{n+2}}_{0} T_n$$

$$+ \sum_{j=2}^{n} w_j \left(T_j - 2x T_{j-1} + T_{j-2} \right)$$

$$= w_0 + w_1 x - 2x w_1 + \sum_{j=2}^{m} w_j \underbrace{\left(T_j - 2x T_{j-1} + T_{j-2} \right)}_{0}$$

$$\boxed{g(x) = w_0 - x w_1}$$

We arrange the data so that all the abscissas $\{x_i\}$ lie in the interval $[-1, 1]$.

If the original data do not satisfy $\min\{x_i\} = -1$

and $\max\{x_n\} = 1$ but instead lie in another interval $[a,b]$ then we can use the change of variable

$$x = \frac{1}{2}(b-a)z + \frac{1}{2}(a+b)$$

to produces a variable z that traverses $[-1,1]$ as x traverses $[a,b]$.