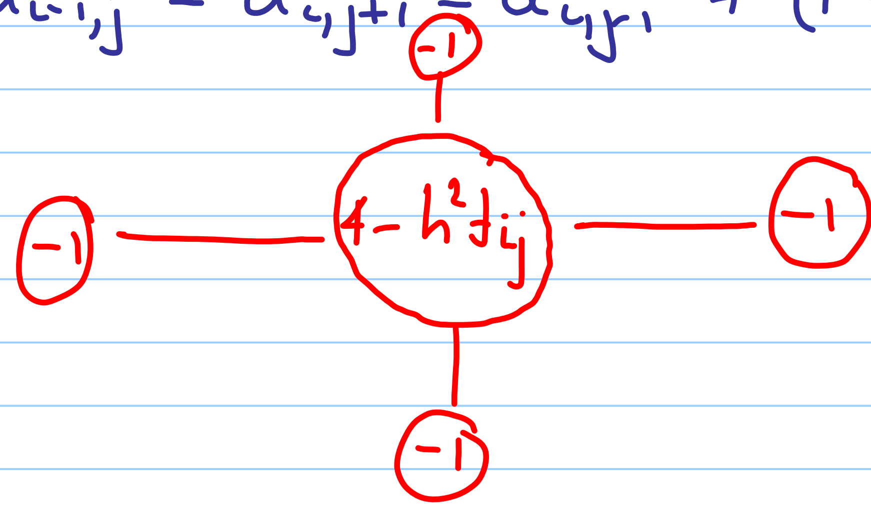
Note Title 12/1/2010

Elliptic Problems

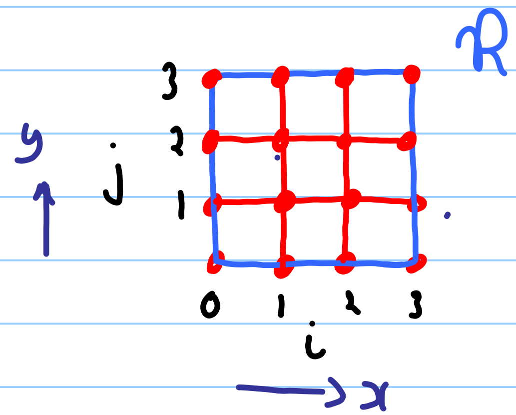
$$\begin{cases} \nabla^2 u + f u = g \\ u(x,y) \text{ known on the boundary } R. \end{cases}$$

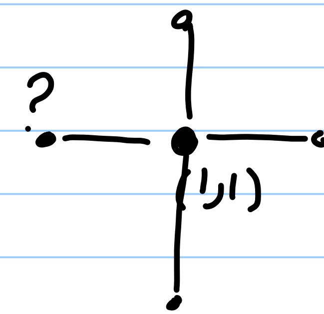
$$-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + (4 - h^2 f_{ij})\, u_{ij} = -h^2 g_{ij}$$



Example

Consider the grid





Solving for the interior nodes

$$-u_{01} - u_{21} - u_{10} - u_{12} + (4 - h^2 f_{11})u_{11} = -h^2 g_{11}$$

$$-u_{11} - u_{31} - u_{20} - u_{22} + (4 - h^2 f_{21})u_{21} = -h^2 g_{21}$$

$$-u_{02} - u_{22} - u_{11} - u_{13} + (4 - h^2 f_{12}) u_{12} = -h^2 g_{12}$$

$$-u_{12} - u_{32} - u_{21} - u_{23} + (4 - h^2 f_{22}) u_{22} = -h^2 g_{22}$$

Rewrite as

$$\begin{cases} (4 - h^2 f_{11}) u_{11} - u_{21} - u_{12} = -h^2 g_{11} + u_{01} + u_{10} \\ -u_{11} + (4 - h^2 f_{21}) u_{21} - u_{22} = -h^2 g_{21} + u_{20} + u_{31} \\ -u_{11} + (4 - h^2 f_{12}) u_{12} - u_{22} = -h^2 g_{12} + u_{02} + u_{13} \\ -u_{12} - u_{21} + (4 - h^2 f_{22}) u_{22} = -h^2 g_{22} + u_{32} + u_{23} \end{cases}$$

Writing in matrix form

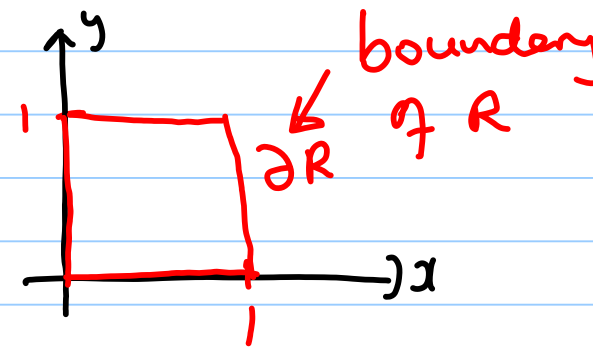
$$
\underbrace{\begin{bmatrix} 4-h^2 f_{11} & -1 & -1 & 0 \\ -1 & 4-h^2 f_{21} & 0 & -1 \\ -1 & 0 & 4-h^2 f_{12} & -1 \\ 0 & -1 & -1 & 4-h^2 f_{22} \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix}}_{u}
=
\underbrace{\begin{bmatrix} -h^2 g_{11} + u_{01} + u_{10} \\ -h^2 g_{21} + u_{20} + u_{31} \\ -h^2 g_{12} + u_{02} + u_{13} \\ -h^2 g_{22} + u_{32} + u_{23} \end{bmatrix}}_{b}
$$

$\boxed{Au = b}$ system to be solved.

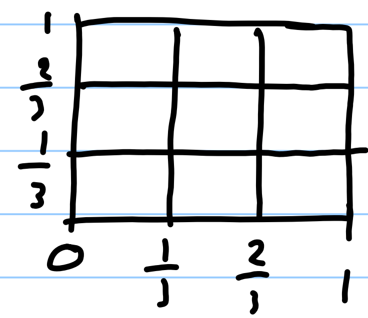
Example

$\nabla^2 u = 9(x^2 + y^2) \qquad 0 < x < 1 \qquad 0 < y < 1$

$u = x - y \qquad (x, y) \in \partial R$



using $h = \frac{1}{3}$



$f_{11} = f(x_1, y_1) = 0$

$f_{ij} = 0$ for all $i, j = 1, 2.$

$g_{11} = g(x_1, y_1) = 9(x_1^2 + y_1^2) = 9\left(\frac{1}{9} + \frac{1}{9}\right) = 2$

⋮

$$g_{21} = g(x_2, y_1) = 9\left(\frac{4}{9} + \frac{1}{9}\right) = 5$$

$$g_{12} = g(x_1, y_2) = 9\left(\frac{1}{9} + \frac{4}{9}\right) = 5$$

$$g_{22} = g(x_2, y_2) = 9\left(\frac{4}{9} + \frac{4}{9}\right) = 8$$

$$u(0,1) = u(x_0, y_1)$$
$$= x_0 - y_1$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & -1 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -\frac{1}{9}(2) + \left(-\frac{1}{3}\right) + \left(\frac{1}{3}\right) \\ \\ \\ \end{bmatrix}$$