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Note Title 8/27/2010

Successive overrelaxation (SOR) method

The SOR method is similar to the Jacobi and Gauss-Seidel methods, but it uses a scaling factor to more rapidly reduce the approximation error. A relaxation factor ω is introduced in the following manner.

$$x_i^{(k)} = \omega\left[-\sum_{\substack{j=1\\ j<i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)} - \sum_{\substack{j=1\\ j>i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k-1)} + \frac{b_i}{a_{ii}}\right]$$

$$+ (1-\omega)\, x_i^{(k-1)} \qquad 1 \le i \le n$$

Note that SOR method with $\omega = 1$ reduces to the Gauss-Seidel method.

<u>Example</u>

Solve $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$ using SOR method

with $\omega = 1.1$ and $x^{(0)} = [0 \quad 0 \quad 0]^T$

$$x_1^{(k)} = \omega\left[-\frac{(-1)}{2}x_2^{(k-1)} + \frac{1}{2}\right] + (1-\omega)\,x_1^{(k-1)}$$

$$x_2^{(k)} = \omega\left[-\frac{(-1)}{3}x_1^{(k)} - \frac{(-1)}{3}x_3^{(k-1)} + \frac{8}{3}\right] + (1-\omega)\,x_2^{(k-1)}$$

$$x_3^{(k)} = \omega\left[-\frac{(-1)}{2}x_2^{(k)} + \frac{-5}{2}\right] + (1-\omega)\,x_3^{(k-1)}$$

that is

$$x_1^{(k)} = \omega\left[\frac{1}{2}x_2^{(k-1)} + \frac{1}{2}\right] + (1-\omega)\,x_1^{(k-1)}$$

$$x_2^{(k)} = \omega\left[\frac{1}{3}x_1^{(k)} + \frac{1}{3}x_3^{(k-1)} + \frac{8}{3}\right] + (1-\omega)\,x_2^{(k-1)}$$

$$x_3^{(k)} = \omega\left[\frac{1}{2}x_2^{(k)} - \frac{5}{2}\right] + (1-\omega)\,x_3^{(k-1)}$$

$$x_1^{(1)} = \omega\left[\frac{1}{2}x_2^{(0)} + \frac{1}{2}\right] + (1-\omega)\,x_1^{(0)}$$

$$= 1.1\left[\frac{1}{2}(0) + \frac{1}{2}\right] + (-0.1)(0)$$

$$= 0.55$$

$$x_2^{(1)} = \omega\left[\frac{1}{3}x_1^{(1)} + \frac{1}{3}x_3^{(0)} + \frac{8}{3}\right] + (1-\omega)\,x_2^{(0)}$$

$$= 1.1\left[\frac{1}{3}(0.55) + \frac{1}{3}(0) + \frac{8}{3}\right] + (-0.1)(0)$$

$$= 3.135$$

$$x_3^{(1)} = \omega\left[\frac{1}{2}x_2^{(1)} - \frac{5}{2}\right] + (1-\omega)\,x_3^{(0)}$$

$$= 1.1\left[\frac{1}{2}(3.135) - \frac{5}{2}\right] + (-0.1)(0)$$

$$= -1.02575$$

$$x^{(1)} = \begin{bmatrix} 0.55 & 3.135 & -1.02575 \end{bmatrix}^T$$

$$x_1^{(2)} = \omega\left[\frac{1}{2}x_2^{(1)} + \frac{1}{2}\right] + (1-\omega)\,x_1^{(1)}$$

$$= 1.1\left[\frac{1}{2}(3.135) + \frac{1}{2}\right] + (-0.1)(0.55)$$

$$= 2.2193$$

$$x_2^{(2)} = \omega\left[\frac{1}{3}x_1^{(2)} + \frac{1}{3}x_3^{(1)} + \frac{8}{3}\right] + (1-\omega)\,x_2^{(1)}$$

$$x_3^{(2)} = \omega\left[\frac{1}{2}x_2^{(2)} - \frac{5}{2}\right] + (1-\omega)\,x_3^{(1)}$$

$\boxed{\#3}$ Hw1

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 = \frac{1}{63}$$

$$\frac{1}{3}x_1 + \frac{1}{4}x_2 = \frac{1}{168}$$

$$x = \begin{bmatrix} \frac{1}{7} \\ -\frac{1}{6} \end{bmatrix} \qquad \tilde{x} = \begin{bmatrix} 0.142 \\ -0.166 \end{bmatrix}$$

$$\|A\tilde{x} - b\|_\infty = ?$$

$$A\tilde{x} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0.142 \\ -0.166 \end{bmatrix} = \begin{bmatrix} 0.0157 \\ 0.0058 \end{bmatrix}$$

$$A\tilde{x} - b \simeq \begin{bmatrix} 0.0157 \\ 0.0058 \end{bmatrix} - \begin{bmatrix} \frac{1}{63} \\ 1/168 \end{bmatrix} \cong \begin{bmatrix} -2.06349 \times 10^{-4} \\ -1.19047 \times 10^{-4} \end{bmatrix}$$

$$\|A\hat{x} - b\|_\infty = \max\left\{ \left|-2.06349 \times 10^{-4}\right|, \left|-1.19047 \times 10^{-4}\right| \right\}$$

$$\approx 2.06 \times 10^{-4}$$