Note Title 8/30/2010

August 30, 2010

Convergence Theorems

Problem Solve a non singular system $Ax = b$ iteratively given an initial vector $x^{(0)}$.

Recall the recursive equation

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b \qquad k = 1, 2, 3, \ldots$$

i.e.

$$x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$$

$$x^{(k)} = G x^{(k-1)} + h$$

where $G = I - Q^{-1}A$ is the iteration matrix

$h = Q^{-1}b$ is the iteration vector

Even though we generally do not compute Q^{-1} when solving the system iteratively, Q^{-1} is usefull in the analysis of convergence of the method.

Let x be the solution of the above system

$$Ax = b.$$

[matrix A is non singular so x exists and it is unique!

$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

Uniqueness: Suppose x & y are solutions

$Ax = b$ & $Ay = b$

$\Rightarrow Ax = Ay$

$A(x-y) = 0$

A non singular

$x - y = 0$

Subtract x from the recursive equation

$$\begin{aligned} x^{(k)} - x &= (I - Q^{-1}A)x^{(k-1)} - x + Q^{-1}b \\ &= (I - Q^{-1}A)x^{(k-1)} - x + Q^{-1}Ax \\ &= (I - Q^{-1}A)x^{(k-1)} - (I - Q^{-1}A)x \end{aligned}$$

$$\Rightarrow x^{(k)} - x = (I - Q^{-1}A)\left[x^{(k-1)} - x\right] \quad (*)$$

Let $e^{(k)} \equiv x^{(k)} - x$ i.e. the error vector at kth iteration

From equation $(*)$ we get

$$e^{(k)} = (I - Q^{-1}A)\, e^{(k-1)}$$

Wish: $e^{(k)}$ to become smaller as k increases

Question: When is $e^{(k)}$ smaller than $e^{(k-1)}$?

- If $(I - Q^{-1}A)$ is small (in some sense)
- would need $Q^{-1}A$ close to I.

- thus Q should be 'close' to A.

<u>Theorem 1</u> Spectral Radius Theorem

In order that the sequence generated by

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b$$

to converge, no matter what the initial vector $x^{(0)}$ is selected, it is <u>necessary</u> and <u>sufficient</u> that all eigenvalues of $I - Q^{-1}A$ lie in the open unit disc, $|z| < 1$, in the complex plane.

that is

$$\rho(I - Q^{-1}A) < 1 \qquad \text{the spectral radius.}$$

Recall:

For any $n \times n$ matrix, with eigenvalues λ_i

$$\rho(G) = \max_{1 \le i \le n} |\lambda_i|$$

Sequence converges if and only if $\rho(I - Q^{-1}A) < 1$

necessary & suff

Example

Consider $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$

Determine whether the Jacobi, Gauss-Seidel and SOR methods (with $\omega = 1.1$) converge for all initial iterates.

<u>Solution</u>

Jacobi method.

$$B = I - Q^{-1}A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\det(B - \lambda I) = 0$$

$$\det(B - \lambda I) = \det\left(\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right)$$

$$= \det \begin{bmatrix} -\lambda & \frac{1}{2} & 0 \\ \frac{1}{3} & -\lambda & \frac{1}{3} \\ 0 & \frac{1}{2} & -\lambda \end{bmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & \frac{1}{3} \\ \frac{1}{2} & -\lambda \end{vmatrix} - \frac{1}{2} \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & -\lambda \end{vmatrix} + 0 \begin{vmatrix} \frac{1}{3} & -\lambda \\ 0 & \frac{1}{2} \end{vmatrix}$$

$$= -\lambda\left(\lambda^2 - \frac{1}{6}\right) - \frac{1}{2}\left(-\frac{1}{3}\lambda - 0\right) + 0$$

$$= -\lambda^3 + \frac{1}{6}\lambda + \frac{1}{6}\lambda = 0$$

$$\lambda\left(-\lambda^2 + \frac{2}{6}\right) = 0$$

$$\lambda = 0 \quad \text{or} \quad \lambda^2 = \frac{1}{3}$$

$$\lambda = \pm\sqrt{\frac{1}{3}} \approx \pm 0.5774$$

$$\rho(B) = \max\left\{0, \pm\sqrt{1/3}\right\}$$

$$= \sqrt{\frac{1}{3}} < 1$$

By the spectral radius theorem, the Jacobi method converges for any initial iterate.

Gauss-Seidel method:

$$\mathcal{L} = I - Q^{-1}A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{6} \end{bmatrix}$$

Find eigenvalues λ

$$\det(\mathcal{L} - \lambda I) = \det \begin{bmatrix} -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{6}-\lambda & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{6}-\lambda \end{bmatrix}$$

$$= -\lambda \begin{vmatrix} \frac{1}{6}-\lambda & \frac{1}{3} \\ \frac{1}{12} & \frac{1}{6}-\lambda \end{vmatrix}$$

$$= -\lambda\left(\left(\frac{1}{6}-\lambda\right)^2 - \frac{1}{36}\right) = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \left(\frac{1}{6}-\lambda\right)^2 - \frac{1}{36} = 0$$

$$\left(\frac{1}{6}-\lambda\right)^2 = \frac{1}{36}$$

$$\frac{1}{6}-\lambda = \pm\frac{1}{6}$$

$$\frac{1}{6} \pm \frac{1}{6} = \lambda$$

$$\frac{1}{3}, 0 = \lambda$$

$$\rho(A) = \max\left\{0, 0, \frac{1}{3}\right\} = \frac{1}{3} < 1$$

Hence Gauss-Seidel method will converge for any starting vector by the spectral radius theorem

Generally computing $\rho(I - Q^{-1}A)$ may be difficult.

<u>Recall</u>

A general matrix $A = (a_{ij})_{n \times n}$ is <u>strictly diagonally dominant</u> if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad (1 \leq i \leq n)$$