Note Title 8/25/2010

August 25, 2010

Jacobi iteration can be written as

$$x_i^{(k)} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k-1)} + \frac{b_i}{a_{ii}} \qquad 1 \le i \le n$$

Example

Consider the system $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}.$$

using $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ solve the system using Jacobi method.

<u>Solution</u>

$$x_i^{(k)} = -\sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{a_{ij}}{a_{ii}} x_j^{(k-1)} + \frac{b_i}{a_{ii}} \qquad 1 \leq i \leq 3$$

that is

$$\begin{cases} x_1^{(k)} = -\left[\frac{-1}{2} x_2^{(k-1)} + \frac{0}{2} x_3^{(k-1)} \right] + \frac{1}{2} \\ x_2^{(k)} = -\left[\frac{-1}{3} x_1^{(k-1)} + \frac{-1}{3} x_3^{(k-1)} \right] + \frac{8}{3} \end{cases}$$

$$x_3^{(k)} = -\left[\frac{0}{2} x_1^{(k-1)} + \frac{-1}{2} x_2^{(k-1)} \right] + \frac{-5}{2}$$

that is

$$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{3} x_1^{(k-1)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k-1)} - \frac{5}{2} \end{cases}$$

$$x^{(0)} = [\,0 \quad 0 \quad 0\,]^T$$

$$x_1^{(1)} = \frac{1}{2} x_2^{(0)} + \frac{1}{2} = \frac{1}{2}(0) + \frac{1}{2} = \frac{1}{2}$$

$$x_2^{(1)} = \frac{1}{3} x_1^{(0)} + \frac{1}{3} x_3^{(0)} + \frac{8}{3}$$

$$= \frac{1}{3}(0) + \frac{1}{3}(0) + \frac{8}{3} = \frac{8}{3}$$

$$x_3^{(1)} = \frac{1}{2} x_2^{(0)} - \frac{5}{2}$$

$$= \frac{1}{2}(0) - \frac{5}{2} = -5/2$$

So

$$x^{(1)} = \left[\frac{1}{2} \quad \frac{8}{3} \quad \frac{-5}{2}\right]^T$$

Next Iteration

$$x_1^{(2)} = \frac{1}{2}x_2^{(1)} + \frac{1}{2} = \frac{1}{2}\left(\frac{8}{3}\right) + \frac{1}{2} = \frac{11}{6}$$

$$x_2^{(2)} = \frac{1}{3}x_1^{(1)} + \frac{1}{3}x_3^{(1)} + 8/3$$

$$= \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{-5}{2}\right) + \frac{8}{3} = 2$$

$$x_3^{(2)} = \frac{1}{2}x_2^{(1)} - \frac{5}{2} = \frac{1}{2}\left(\frac{8}{3}\right) - \frac{5}{2} = -7/6$$

$$x^{(2)} = \begin{bmatrix} 11/6 \\ 2 \\ -7/6 \end{bmatrix}$$

The process continues until stopping criterion is satisfied.

Recall the general recursive equation

$$Q x^{(k)} = (Q - A) x^{(k-1)} + b \qquad k = 1, 2, 3, \ldots$$

Note that

$$x^{(k)} = Q^{-1}(Q - A) x^{(k-1)} + Q^{-1} b$$

$$x^{(k)} = (I - Q^{-1}A) x^{(k-1)} + Q^{-1} b$$

In our previous example take Q to be the diagonal of A.

So

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ really nice!

$$Q^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$Q^{-1}A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

The Jacobi iteration matrix and constant vector are

$$B = I - Q^{-1}A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \qquad h = Q^{-1}b = \begin{bmatrix} 1/2 \\ 8/3 \\ -5/2 \end{bmatrix}$$

The Jacobi method in matrix' form becomes

$$x^{(k)} = Bx^{(k-1)} + h$$

$$x^{(k)} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} \frac{1}{2} \\ 8/3 \\ -5/2 \end{bmatrix}$$

$$x^{(1)} = Bx^{(0)} + h$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 8/3 \\ -5/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 8/3 \\ -5/2 \end{bmatrix}$$

$$x^{(2)} = Bx^{(1)} + h$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 8/3 \\ -5/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 8/3 \\ -5/2 \end{bmatrix} =$$

Gauss-Seidel Method

Recall that in the Jacobi method, the equations are solved in order.

In Gauss-Seidel method, the components $x_j^{(k-1)}$ and the corresponding new values $x_j^{(k)}$ are used immediately in their place.

$$x_1^{(k)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k-1)} + a_{13}x_3^{(k-1)} + \dots + a_{1n}x_n^{(k-1)}\right) + \frac{b_1}{a_{11}}$$

know this by now

$$x_2^{(k)} = -\frac{1}{a_{22}}\left[a_{21}x_1^{(k)} + a_{23}x_3^{(k-1)} + \dots + a_{2n}x_n^{(k-1)} \right] + \frac{b_2}{a_{22}}$$

$$x_3^{(k)} = -\frac{1}{a_{33}}\left[a_{31}x_1^{(k)} + a_{32}x_2^{(k)} + a_{34}x_4^{(k-1)} + \dots + a_{3n}x_n^{(k-1)} \right] + \frac{b_3}{a_{33}}$$

$$\vdots$$

$$x_n^{(k)} = -\frac{1}{a_{nn}}\left[a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn-1}x_{n-1}^{(k)} \right] + \frac{b_n}{a_{nn}}$$

Compact form

$$x_i^{(k)} = -\sum_{\substack{j=1 \\ j<i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k-1)} + \frac{b_i}{a_{ii}}$$

<u>Example</u>

$$Ax = b \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$$

$$x^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

$$x_1^{(1)} = -\left[\frac{-1}{2} x_2^{(0)} + \frac{0}{2} x_3^{(0)}\right] + \frac{1}{2}$$

$$= \frac{1}{2} x_2^{(0)} + \frac{1}{2}$$

$$= \frac{1}{2}(0) + \frac{1}{2} = \frac{1}{2}$$

$$x_2^{(1)} = -\left[\frac{-1}{3} x_1^{(1)} + \frac{-1}{3} x_3^{(0)}\right] + \frac{8}{3}$$

$$= \frac{1}{3} x_1^{(1)} + \frac{1}{3} x_3^{(0)} + \frac{8}{3}$$

$$= \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}(0) + \frac{8}{3} = \frac{17}{6}$$

$$x_3^{(1)} = -\left[\frac{0}{2}x_1^{(1)} + \frac{-1}{2}x_2^{(1)}\right] + \frac{-5}{2}$$

$$= \frac{1}{2}x_2^{(1)} + \frac{-5}{2}$$

$$= \frac{1}{2}\left(\frac{17}{6}\right) - 5/2$$

$$= -\frac{13}{12}$$