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Note Title 8/20/2010

Definition

For an $n \times n$ matrix, we define the matrix norm $\|A\|$ subordinate to a given vector norm to be

$$\|A\| = \sup \left\{ \|Ax\| : x \in \mathbb{R}^n \text{ and } \|x\| = 1 \right\}$$

Sup – Supremum
least upper bound

$$= \sup_{\|x\|=1} \|Ax\|$$

The supremum is taken over all n-dimensional vectors with unit norm.

<u>Lemma</u>

A subordinate matrix norm satisfies the following properties, for any matrices A & B.

(i) $\|A\| > 0$ if $A \neq 0$

(ii) $\|\alpha A\| = |\alpha| \, \|A\|$ for any scaler α.

(iii) $\|A + B\| \leq \|A\| + \|B\|$ triangular inequality

We also have the additional properties

(iv) $\|I\| = 1$ $\qquad I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

(v) $\|Ax\| \leq \|A\| \, \|x\|$

(vi) $\|AB\| \leq \|A\| \, \|B\|$

The matrix norms subordinate to the vector norms $\|\cdot\|_1$, $\|\cdot\|_2$ & $\|\cdot\|_\infty$ are

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad l_1\text{-matrix norm}$$

Sum over the columns.

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \quad \ell_\infty\text{-matrix norm}$$

$$\|A\|_2 = \max_{1 \le i \le n} \sqrt{|\sigma_i|} \quad \text{spectral} / \ell_2\text{-matrix norm}$$

the σ_i are the eigenvalues of $A^T A$ (called the singular values of A).

The largest σ_{max} in absolute value is known as the <u>spectral radius</u> of A.

Example

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$$

To compute l_∞-matrix norm we sum over the rows.

row1: $\sum_{j=1}^{3} a_{1j} = |1| + |2| + |-1| = 4$

row2: $\sum_{j=1}^{3} |a_{2j}| = |0| + |3| + |-1| = 4$

row3: $\sum_{j=1}^{3} |a_{3j}| = |5| + |-1| + |1| = 7$

$$\|A\|_\infty = \max\{4, 4, 7\} = 7$$

To compute l_1- matrix norm sum over columns

$$\sum_{i=1}^{3} |a_{i,1}| = |1| + |0| + |5| = 6$$

$$\sum_{i=1}^{3} |a_{i,2}| = |2| + |3| + |-1| = 6$$

$$\sum_{i=1}^{3} |a_{i,3}| = |-1| + |-1| + |1| = 3$$

$$\|A\|_1 = \max\{6, 6, 3\} = 6$$

Condition Number and Ill-Conditioning

Definition

We define the condition number of an $n \times n$ non singular matrix A for the norm $\|\cdot\|_p$ to be

$$\kappa_p(A) = \|A\|_p \, \|A^{-1}\|_p$$

The condition number of a matrix A gives a measure of how sensitive systems of equations, with coefficient matrix A, are to small perturbations.

If a system $Ax = b$ is sensitive to perturbations then the matrix A is said to be ill-conditioned and the condition number is large.

Suppose we want to solve $Ax = b$ where A is non singular.

Suppose the right side is perturbed by an amount δb and the solution is perturbed by an amount δx

Then

$$A(x + \delta x) = Ax + A\,\delta x$$

$$= b + \underline{\underline{A\,\delta x}}$$

$$= b + \underline{\underline{\delta b}} \quad \text{where } A\,\delta x = \delta b$$

Recall that for $Ax = b$ we have

$$\|Ax\| = \|b\|$$

So

$$\|b\| = \|Ax\| \le \|A\|\,\|x\|$$

i.e.

$$\|b\| \le \|A\|\,\|x\|$$

$$\Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|} \quad (*)$$

For the perturbed linear problem

$$A\,\delta x = \delta b$$

$$A\,\delta x = \delta b$$
$$\underbrace{A^{-1}A}\,\delta x = A^{-1}\delta b$$
$$I\,\delta x = A^{-1}\delta b$$
$$\delta x = A^{-1}\delta b$$

we have

$$\delta x = A^{-1}\delta b$$

because A is non singular $\therefore$ it is invertible

So that

$$\|\delta x\| = \|A^{-1}\delta b\|$$
$$\le \|A^{-1}\|\,\|\delta b\|$$

i.e.

$$\|\delta x\| \le \|A^{-1}\|\,\|\delta b\| \quad (**)$$

Combining (*) and (**)

$$\frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$$

$$\|\delta x\| \le \|A^{-1}\| \, \|\delta b\|$$

we get $$\frac{\|\delta x\|}{\|x\|} \le \|A\| \, \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}$$

i.e.

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A) \frac{\|\delta b\|}{\|b\|}$$ where $\kappa(A)$ is the condition number of A