Wednesday 18 2010

Note Title 8/18/2010

8.2 Iterative solution of Linear systems.

We want to solve a non singular system of the form

$$Ax = b.$$

Example

$$2x + y = 3$$
$$5x - 2y = 7$$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 7 \end{bmatrix}}_{b}$$

Vector and Matrix Norms

Norms can be defined on any vector space.
but we shall restrict ourselves to $\mathbb{R}^n$.

$\mathbb{R}^2$: $\quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

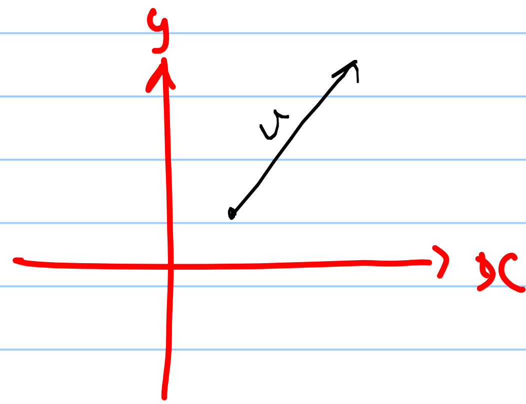
$\mathbb{R}^3$: $\quad w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad z = \begin{bmatrix} .5 \\ 0 \\ 1 \end{bmatrix}$

$\mathbb{R}^n$: $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

We can think of a vector norm $||x||$ as the <u>length</u> or <u>magnitude</u> of a vector $x \in \mathbb{R}^n$.

in
member of

$\mathbb{R}^2$



$||u|| =$ length

A vector norm is any mapping from $\mathbb{R}^n$ to $\mathbb{R}$ that satisfy the three properties

① $\|x\| > 0$ if $x \neq 0$ $\quad \|x\| = 0$ if $x = 0$

② $\|\alpha x\| = |\alpha| \, \|x\|$

③ $\|x + y\| \leq \|x\| + \|y\|$ triangle inequality

for any vectors $x, y \in \mathbb{R}^n$ and scalars $\alpha \in \mathbb{R}$.

$$\|\cdot\| : \underset{\text{vector}}{\mathbb{R}^n} \longrightarrow \underset{\text{number}}{\mathbb{R}}$$

Examples

Euclidean norm (l_2-vector norm or length)

$$\|x\|_2 = \left(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right)^{1/2}$$

$$= \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$= \left(x^T x\right)^{1/2}$$

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad u^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$u^T u = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= (1)(1) + (2)(2)$$

ℓ_1-vector norm:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$$= \sum_{i=1}^{n} |x_i|$$

ℓ_∞-vector norm:

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

In general l_p-norm $\quad p \geq 1 \quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

<u>Examples</u>

Let
$$x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$$

Then

$$\|x\|_1 = |1| + |-2| + |3| = 6$$

$$\|x\|_2 = \left(1^2 + |-2|^2 + 3^2\right)^{1/2}$$
$$= (1 + 4 + 9)^{1/2}$$
$$= \sqrt{14}$$

$$\|x\|_\infty = \max\{|1|, |-2|, |3|\}$$

$$= \max\{1, 2, 3\}$$

$$= 3$$

$$\|y\|_1 = 0 + 2 + 3 = 5$$

$$\|y\|_2 = (0^2 + 2^2 + 3^2)^{1/2} = \sqrt{13}$$

$$\|y\|_\infty = \max\{0, 2, 3\} = 3$$

Example

Let $u = [1 \ 1 \ 1 \ 1 \ 1]^T$ &

$v = [0.1 \ 0.7 \ 0.3 \ 0.4 \ 0.5]^T$

$\|u\|_2 = \sqrt{5}$ $\quad \|v\|_2 = \left(0.1^2 + 0.7^2 + 0.3^2 + .4^2 + .5^2\right)^{1/2}$

$\|u\|_1 = 5$ $\quad \|v\|_1 =$

$\|u\|_\infty = 1$ $\quad \|v\|_\infty = 0.7$

Two different vectors may have the same norm so

$$\|x\| = \|y\| \not\Rightarrow x = y$$

However

$$\|x - y\| = 0 \Rightarrow x - y = 0 \quad \text{by property of norms}$$

$$\Rightarrow x = y$$