

Name : \_\_\_\_\_

1. Given the linear system of equations

$$\begin{cases} 2x_1 - 6\alpha x_2 = 3 \\ 3\alpha x_1 - x_2 = \frac{3}{2} \end{cases}$$

Find value(s) of  $\alpha$  for which the system has no solutions. Find value(s) of  $\alpha$  for which the system has an infinite number of solutions. Assuming a unique solution exists for a given alpha, find the solution.

2. For what values of  $\alpha$  does naive Gaussian elimination produce erroneous answers for this system

$$\begin{cases} x_1 + x_2 = 2 \\ \alpha x_1 + x_2 = 2 + \alpha \end{cases}$$

Explain what happens in the computer.

3. Show that the following equations are consistent if and only if,  $a = +1$  or  $a = -1$ .

$$\begin{cases} x + y + z = 1 + a^2 \\ x + 2y + 3z = -2a \\ x + 3y + 4z = -4a \\ x + 2y + 2z = 2(1 - a) \end{cases}$$

4. Apply naive Gaussian elimination to the systems of equations below and account for the failures.

$$\bullet \begin{cases} x_1 + x_2 + 2x_3 = 4 \\ x_1 + x_2 = 2 \\ x_2 + x_3 = 0 \end{cases}$$

$$\bullet \begin{cases} 0x_1 + 2x_2 = 4 \\ x_1 - x_2 = 5 \end{cases}$$

5. Consider

$$\mathbf{A} = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix}.$$

Compute residual vectors  $\tilde{\mathbf{r}} = \mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}$  and  $\hat{\mathbf{r}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{b}$  and decide which of  $\tilde{\mathbf{x}}$  and  $\hat{\mathbf{x}}$  is the better solution vector. Now compute the error vectors  $\tilde{\mathbf{e}} = \tilde{\mathbf{x}} - \mathbf{x}$  and  $\hat{\mathbf{e}} = \hat{\mathbf{x}} - \mathbf{x}$ , where  $\mathbf{x} = [1, -1]^T$  is the exact solution. Discuss the implications of this example.

6. Solve the system below using naive Gaussian elimination.

$$\begin{cases} 3x_1 + 2x_2 - 5x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \\ x_1 + 4x_2 - x_3 = 4 \end{cases}$$

7. Consider the systems of equations

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 6 \\ x_1 + 3x_2 + 4x_3 = 0 \\ 5x_1 + 2x_2 = 2 \end{cases}$$

and

$$\begin{cases} 2x_1 + 3x_2 = 8 \\ -x_1 + 2x_2 - x_3 = 0 \\ 3x_1 + 2x_3 = 9 \end{cases}.$$

In each case, solve for  $x_1$ ,  $x_2$  and  $x_3$  using Gaussian elimination with partial pivoting. Show all intermediate matrices and vectors.

8. Prove that

- $2^n < n!$  for every positive integer  $n$  with  $n \geq 4$ .

- $\sum_{k=1}^n 2^k = 2^{n+1} - 2$

9. How many storage locations are needed for a system of  $n$  linear equations if the coefficient matrix has banded structure in which  $a_{ij} = 0$  for  $|i - j| \geq k + 1$ ?

10. What is the appearance of a matrix  $A$  if its elements satisfy  $a_{ij} = 0$  when

- $j < i - 2$

- $j > i + 1$

11. Determine whether the matrices below are symmetric and strictly diagonally dominant.

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$$