Name : ______________________________

1. Given the linear system of equations

$$\begin{cases} 2x_1 - 6\alpha x_2 = 3 \\ 3\alpha x_1 - x_2 = \frac{3}{2} \end{cases}$$

   Find value(s) of α for which the system has no solutions. Find value(s) of α for which the system has an infinite number of solutions. Assuming a unique solution exists for a given alpha, find the solution.

2. For what values of α does naive Gaussian elimination produce erroneous answers for this system

$$\begin{cases} x_1 + x_2 = 2 \\ \alpha x_1 + x_2 = 2 + \alpha \end{cases}$$

   Explain what happens in the computer.

3. Show that the following equations are consistent if and only if, $a = +1$ or $a = -1$.

$$\begin{cases} x + y + z = 1 + a^2 \\ x + 2y + 3z = -2a \\ x + 3y + 4z = -4a \\ x + 2y + 2z = 2(1 - a) \end{cases}$$

4. Apply naive Gaussian elimination to the systems of equations below and account for the failures.

- $$\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ x_1 + x_2 = 2 \\ x_2 + x_3 = 0 \end{cases}$$

- $$\begin{cases} 0x_1 + 2x_2 = 4 \\ x_1 - x_2 = 5 \end{cases}$$

5. Consider

$$\mathbf{A} = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}, \qquad \tilde{\mathbf{x}} = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}, \qquad \hat{\mathbf{x}} = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix}.$$

Compute residual vectors $\tilde{r} = A\tilde{x} - b$ and $\hat{r} = A\hat{x} - b$ and decide which of $\tilde{x}$ and $\hat{x}$ is the better solution vector. Now compute the error vectors $\tilde{e} = \tilde{x} - x$ and $\hat{e} = \hat{x} - x$, where $x = [1, -1]^T$ is the exact solution. Discuss the implications of this example.

6. Solve the system below using naive Gaussian elimination.

$$\begin{cases} 3x_1 + 2x_2 - 5x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \\ x_1 + 4x_2 - x_3 = 4 \end{cases}$$

7. Consider the systems of equations

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 6 \\ x_1 + 3x_2 + 4x_3 = 0 \\ 5x_1 + 2x_2 = 2 \end{cases}$$

and

$$\begin{cases} 2x_1 + 3x_2 = 8 \\ -x_1 + 2x_2 - x_3 = 0 \\ 3x_1 + 2x_3 = 9 \end{cases}.$$

In each case, solve for x_1, x_2 and x_3 using Gaussian elimination with partial pivoting. Show all intermediate matrices and vectors.

8. Prove that
   - $2^n < n!$ for every positive integer n with $n \geq 4$.
   - $\displaystyle\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$

9. How many storage locations are needed for a system of n linear equations if the coefficient matrix has banded structure in which $a_{ij} = 0$ for $|i - j| \geq k + 1$?

10. Give an example of a system of linear equations in tridiagonal form that cannot be solved without pivoting.

11. What is the appearance of a matrix A if its elements satisfy $a_{ij} = 0$ when
    - $j < i - 2$
    - $j > i + 1$

12. Determine whether the matrices below are symmetric and strictly diagonally dominant.

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$$