Name : ______________________________

1. Show that the system of equations

$$\begin{cases} x_1 + 4x_2 + \alpha x_3 = 6 \\ 2x_1 - x_2 + 2\alpha x_3 = 3 \\ \alpha x_1 + 3x_2 + x_3 = 5 \end{cases}$$

possesses a unique solution when $\alpha = 0$, no solution when $\alpha = -1$, and infinitely many solutions when $\alpha = 1$.

2. Apply naive Gaussian elimination to the systems of equations below and account for the failures.

- $\begin{cases} 3x_1 + 2x_2 = 4 \\ -x_1 - \frac{2}{3}x_2 = 1 \end{cases}$
- $\begin{cases} 6x_1 - 3x_2 = 6 \\ -2x_1 + x_2 = -2 \end{cases}$
- $\begin{cases} 0x_1 + 2x_2 = 4 \\ x_1 - x_2 = 5 \end{cases}$

3. Consider

$$\mathbf{A} = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix}.$$

Compute residual vectors $\tilde{r} = A\tilde{x} - b$ and $\hat{r} = A\hat{x} - b$ and decide which of $\tilde{x}$ and $\hat{x}$ is the better solution vector. Now compute the error vectors $\tilde{e} = \tilde{x} - x$ and $\hat{e} = \hat{x} - x$, where $x = [1, -1]^T$ is the exact solution. Discuss the implications of this example.

4. Solve the system below using naive Gaussian elimination.

$$\begin{cases} 3x_1 + 2x_2 - 5x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \\ x_1 + 4x_2 - x_3 = 4 \end{cases}$$

5. Consider the systems of equations

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 6 \\ x_1 + 3x_2 + 4x_3 = 0 \\ 5x_1 + 2x_2 = 2 \end{cases}$$

and

$$\begin{cases} 2x_1 + 3x_2 = 8 \\ -x_1 + 2x_2 - x_3 = 0 \\ 3x_1 + 2x_3 = 9 \end{cases}.$$

In each case, solve for x_1, x_2 and x_3 using Gaussian elimination with partial pivoting. Show all intermediate matrices and vectors.

6. Derive the formulae

- $\displaystyle\sum_{k=1}^{n} k = \frac{n}{2}(n+1)$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$

7. How many storage locations are needed for a system of n linear equations if the coefficient matrix has banded structure in which $a_{ij} = 0$ for $|i-j| \geq k+1$?

8. Give an example of a system of linear equations in tridiagonal form that cannot be solved without pivoting.

9. What is the appearance of a matrix A if its elements satisfy $a_{ij} = 0$ when

- $j < i - 2$
- $j > i + 1$

10. Determine whether the matrices below are symmetric and strictly diagonally dominant.

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$$