Trigonometric Functions

6.8 Graphs of Sine and Cosine Functions

November 3, 2010

Periodic Functions

A function f is called a **periodic function** if there is a positive number p such that

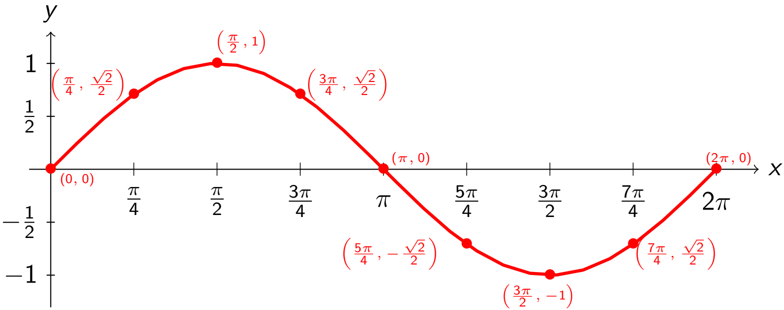
$$f(x+p) = f(x) \qquad \text{for all } x \text{ in the domain of } f.$$

If p is the smallest such number for which this equation holds, then p is called the **fundamental period**.

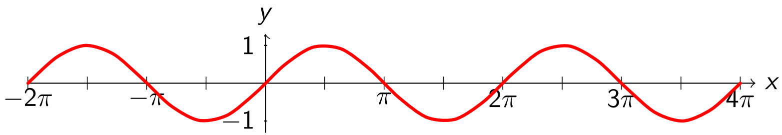
The table below gives the values of the sine function at some of the special angles.

$\mathbf{x}$	$\mathbf{f}(\mathbf{x}) = \sin \mathbf{x}$	$(\mathbf{x}, \mathbf{y})$
0	$\sin 0 = 0$	$(0,\ 0)$
$\frac{\pi}{4}$	$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	$\sin \frac{\pi}{2} = 1$	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	$\sin \pi = 0$	$(\pi, 0)$
$\frac{5\pi}{4}$	$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	$\sin \frac{3\pi}{2} = -1$	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$\sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	$\sin 2\pi = 0$	$(2\pi, 0)$

Plot the above coordinates (x, y) to get the graph of one **period** or **cycle** of the graph of $y = \sin x$.



We can extend the graph horizontally in both directions since the domain of the sine function is the set of all real numbers.



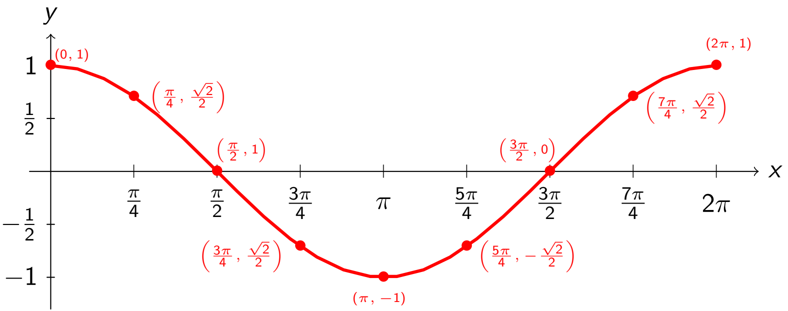
Sine Function $f(x) = \sin x$

- Domain: $(-\infty, \infty)$ or $-\infty \leq x \leq \infty$
- Range: $[-1, 1]$ or $-1 \leq y \leq 1$
- The sine function is an odd function:
  - symmetric about the origin
  - $f(-x) = -f(x)$
- The sine function is a periodic function with fundamental period 2π.
- The x-intercepts, $0, \pm\pi, \pm 2\pi, ...$, are of the form $n\pi$, where n is an integer.
- The maximum (1) and minimum (-1) values of the sine function corresponds to x-values of the form $\dfrac{(2n+1)\pi}{2}$ such as $\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, ...$

The table below gives the values of the cosine function at some of the special angles.

$\mathbf{x}$	$\mathbf{f}(\mathbf{x}) = \cos \mathbf{x}$	$(\mathbf{x}, \mathbf{y})$
0	$\cos 0 = 1$	$(0, 1)$
$\frac{\pi}{4}$	$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	$\cos \frac{\pi}{2} = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	$\cos \pi = -1$	$(\pi, -1)$
$\frac{5\pi}{4}$	$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	$\cos \frac{3\pi}{2} = 0$	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	$\sin 2\pi = 1$	$(2\pi, 1)$

Plot the above coordinates (x, y) to get the graph of one **period** or **cycle** of the graph of $y = \cos x$.



Cosine Function $f(x) = \cos x$

- Domain: $(-\infty, \infty)$ or $-\infty \leq x \leq \infty$
- Range: $[-1, 1]$ or $-1 \leq y \leq 1$
- The cosine function is an even function:
  - symmetric about the y-axis
  - $f(-x) = f(x)$
- The sine function is a periodic function with fundamental period 2π.
- The x-intercepts, $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$ are odd integer multiples of $\frac{\pi}{2}$ that have the form $\dfrac{(2n+1)\pi}{2}$, where n is an integer .
- The maximum (1) and minimum (-1) values of the cosine function corresponds to x-values of the form $n\pi$ such as $0, \pm\pi, \pm 2\pi, ...,$

Plot the functions $y = 2\sin x$ and $y = \frac{1}{2}\sin x$ on the same graph with $y = \sin x$ on the interval $0 \leq x \leq 2\pi$.

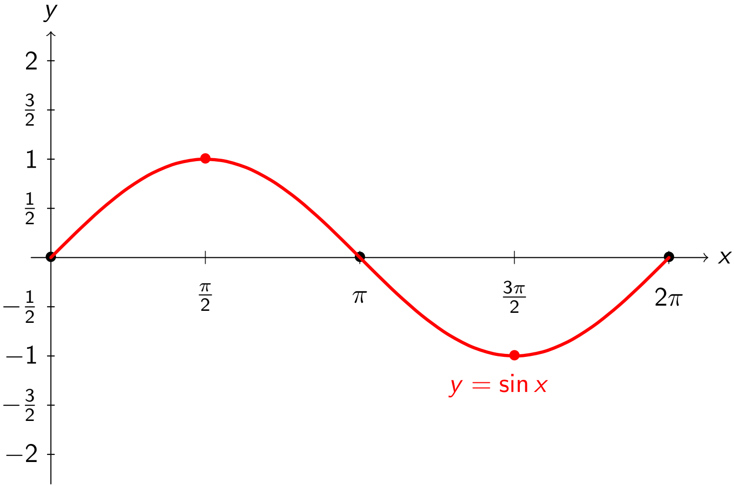
Make a table with the coordinate values of the graphs.

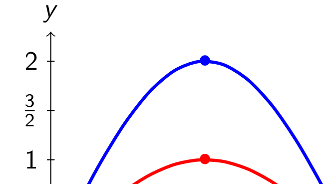
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$					
$2\sin x$					
$\frac{1}{2}\sin x$					

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
$2\sin x$					
$\frac{1}{2}\sin x$					

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
$2\sin x$	0	2	0	-2	0
$\frac{1}{2}\sin x$					

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π


y
2
3/2
1
1/2
−1/2
−1
−3/2
−2
x
π/2
π
3π/2
2π
y = sin x


y
2
3/2
1

Amplitude of Sinusoidal Functions

For sinusoidal functions of the form $y = A \sin Bx$ and $y = A \cos Bx$, the **amplitude** is $|A|$. When $|A| < 1$, the graph is compressed in the vertical direction, and when $|A| > 1$, the graph is stretched in the vertical direction.

Example

State the amplitude of

(a) $y = \frac{2}{3} \sin 4x$
(b) $y = -\cos 7x$
(c) $y = 4 \cos \left(\frac{\pi}{4} x\right)$

Plot the functions $y = \cos 2x$ and $y = \cos(\frac{1}{2}x)$ on the same graph with $y = \cos x$ on the interval $0 \leq x \leq 2\pi$.

Make a table with the coordinate values of the graphs.

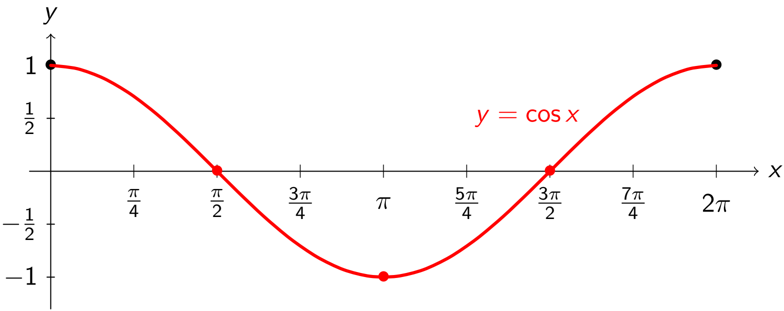
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos x$									
$\cos 2x$									
$\cos(\frac{1}{2}x)$									

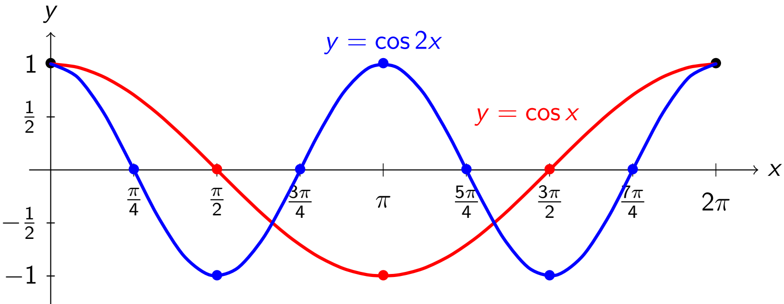
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos x$	1		0		-1		0		1
$\cos 2x$									
$\cos(\frac{1}{2}x)$									

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos x$	1		0		-1		0		1
$\cos 2x$	1	0	-1	0	1	0	-1	0	1
$\cos(\frac{1}{2}x)$									

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

Plot the above coordinates (x, y) to get the graphs.





Period of Sinusoidal Functions

For sinusoidal functions of the form $y = A \sin Bx$ and $y = A \cos Bx$, the **period** is $\frac{2\pi}{B}$. When $0 < B < 1$, the graph is stretched in the horizontal direction, and when $B > 1$, the graph is compressed in the horizontal direction.

Example

State the period of

(a) $y = \frac{2}{3} \sin 4x$

(b) $y = -\cos 7x$

(c) $y = 4 \cos\left(\frac{\pi}{4} x\right)$

Strategy for Sketching Graphs of Sinusoidal Functions

To graph $y = A\sin Bx$ or $y = A\cos Bx$ with $B > 0$:

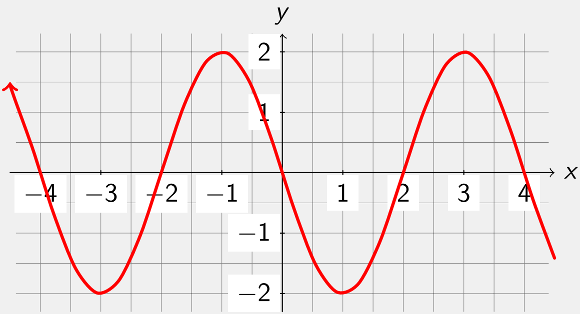
- **Step 1:** Find the amplitude $|A|$ and the period $\frac{2\pi}{B}$
- **Step 2:** Divide the period into four subintervals of equal lengths.
- **Step 3:** Make a table and evaluate the functions for x-values from Step 2 starting at $x = 0$.
- **Step 4:** Draw the xy-plane (label the y-axis from $-|A|$ to $|A|$) and plot the points found in Step 3.
- **Step 5:** Connect the points with a sinusoidal curve (with amplitude $|A|$).
- **Step 6:** Extends the graph over one or two additional periods in both directions (left and right).

Example

Use the strategy for graphing a sinusoidal function to graph $y = -2\sin\left(\frac{1}{4}x\right)$.

Example

Find an equation for the graph.



Consider the shifted sinusoidal functions

$$y = A\sin(Bx + C) + D$$

and

$$y = A\cos(Bx + C) + D$$

where $A \neq 0$, $B \neq 0$, C and D are real numbers. We can write the functions in **standard form** to get

$$y = A\sin\left[B\left(x + \frac{C}{B}\right)\right] + D$$

and

$$y = A\cos\left[B\left(x + \frac{C}{B}\right)\right] + D.$$

The **phase shift** is the quantity $(-\frac{C}{B})$.

Example

State the amplitude, period, and phase shift (including direction) of

(a) $y = 4\cos(x + \pi)$

(b) $y = -7\sin(4x - 3)$

(c) $y = \frac{1}{2}\sin\left(\frac{1}{3}x + \pi\right)$