Polynomial and Rational Functions

4.6 Rational Functions

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Definition: Rational Function

A function $f(x)$ is a **rational function** if

$$f(x) = \frac{n(x)}{d(x)}, \quad d(x) \neq 0$$

where the numerator, $n(x)$, and the denominator, $d(x)$, are polynomial functions. The domain of $f(x)$ is the set of all real numbers x such that $d(x) \neq 0$.

Examples

Find the domain of the rational functions, expressing the domain in interval notation.

$$\textbf{(a).}\ f(x) = \frac{x+1}{x^2 - x - 6} \qquad \textbf{(b).}\ g(x) = \frac{3x}{x^2 + 9}$$

Definition: Vertical Asymptotes

The line $x = a$ is a **vertical asymptote** for the graph of a function if $f(x)$ either increases or decreases without bound as x approaches a from either the left or the right.

Locating Vertical Asymptotes

Let $f(x) = \frac{n(x)}{d(x)}$ be a rational function in lowest terms (i.e. assume $n(x)$ and $d(x)$ are polynomials with no common factors); then the graph of f has a vertical asymptote at any real zero of the denominator $d(x)$.

Examples

Locate any vertical asymptotes of the rational functions

- $f(x) = \dfrac{5x+2}{6x^2 - x - 2}$
- $f(x) = \dfrac{x+2}{x^3 - 3x^2 - 10x}$

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function if $f(x)$ approaches b as x increases or decreases without bound.

Locating Horizontal Asymptotes

Let f be a rational function given by

$$f(x) = \frac{n(x)}{d(x)} = \frac{a_n x^{n} + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^{m} + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

where $n(x)$ and $d(x)$ are polynomials.

- When $n < m$, the x-axis ($y = 0$) is the horizontal asymptote.
- When $n = m$, the line $y = \frac{a_n}{b_n}$ (ratio of leading coefficients) is the horizontal asymptote.
- When $n > m$, there is no horizontal asymptote.

Examples

Determine whether a horizontal asymptote exists for the graph of each of the given rational functions. If it does, locate the horizontal asymptote.

- $f(x) = \dfrac{8x+3}{4x^2+1}$
- $g(x) = \dfrac{8x^2+3}{4x^2+1}$
- $h(x) = \dfrac{8x^3+3}{4x^2+1}$

Slant Asymptote

Let f be a rational function given by $f(x) = \frac{n(x)}{d(x)}$, where $n(x)$ and $d(x)$ are polynomials and the degree of $n(x)$ is *one more than* the degree of $d(x)$. On dividing $n(x)$ by $d(x)$, the rational function can be expressed as

$$f(x) = mx + b + \frac{r(x)}{d(x)}$$

where the degree of the remainder $r(x)$ is less than the degree of $d(x)$ and the line $y = mx + b$ is a **slant asymptote** for the graph of f.

Example

Determine the slant asymptote of the rational function

$$f(x) = \frac{x^2 + 9x + 20}{x - 3}.$$