Polynomial and Rational Functions

4.2 Polynomial Functions of Higher Degree

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Definition: Polynomial Function

Let n be a nonnegative integer, and let $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n**. The coefficient a_n is called the **leading coefficient**, and a_0 is the constant.

Example 1

For each of the given functions, determine whether the function is a polynomial function. If it is a polynomial function, state the degree of the polynomial.

a. $f(x) = 3 - 2x^5$ **b.** $F(x) = \sqrt{x} + 1$
c. $g(x) = 2$ **d.** $h(x) = 3x^2 - 2x + 5$
e. $H(x) = 4x^5(2x-3)^2$ **f.** $G(x) = 2x^4 - 5x^3 - 4x^{-2}$

Graphs of Polynomial Functions

Polynomial	**Degree**	**Special Name**	**Graph**
$f(x) = c$	0	Constant function	Horizontal line
$f(x) = mx + b$	1	Linear function	Line • Slope $= m$ • y-intercept: $(0, b)$
$f(x) = ax^2 + bx + c$	2	Quadratic function	Parabola – Opens • up if $a > 0$ • down if $a < 0$

Graphs of all polynomial functions are both *continuous* and *smooth*.

- A **continuous** graph is one you can draw completely without picking up your pencil (the graph has no jumps or holes).
- A **smooth** graph has no sharp corners.

Definition: Power Function

Let n be a positive integer and the coefficient $a \neq 0$ be a real number. The function

$$f(x) = ax^n$$

is called a **power function of degree** n.

Power functions with *even* powers look similar to the square function.

Power functions with *odd* powers (other than $n = 1$) look similar to the cube function.

Real Zeros of Polynomial Functions

If $f(x)$ is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a **solution**, or **root**, of the equation $f(x) = 0$.
2. $(a, 0)$ is an x-**intercept** of the graph of $f(x)$.
3. $x = a$ is a **zero** of the function $f(x)$.
4. $(x - a)$ is a **factor** of $f(x)$.

Consider the polynomial function $f(x) = x^2 - 1$.

Example 3

Find the real zeros of the polynomial function $f(x) = x^3 + x^2 - 2x$.

Definition: Multiplicity of a Zero

If $(x - a)^n$ is a factor of a polynomial f, then a is called a **zero of multiplicity n** of f.

Example 4

Find the zeros, and state their multiplicities, of the polynomial function $g(x) = (x - 1)^2(x + \frac{3}{5})^7(x + 5)$.

Example 5

Find a polynomial of degree 7 whose zeros are

-2 (multiplicity 2) 0 (multiplicity 4) 1 (multiplicity 1).

Multiplicity of a zero and relation to the graph of a polynomial

If a is zero of $f(x)$, then:

Multiplicity of a	**$f(x)$ on either side of $x = a$**	**Graph of Function at the Intercept**
Even	Does not change sign	Touches the x-axis (turns around) at point $(a, 0)$
Odd	Changes sign	Crosses the x-axis at point $(a, 0)$

End Behavior

As x gets large in the positive ($x \to \infty$) and negative ($x \to -\infty$) directions, the graph of the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

has the same behavior as the power function

$$y = a_n x^n.$$

To graph a polynomial function of degree 3 or greater

1. Determine the y-intercept.
2. Find the zeros of the polynomial (note the multiplicities).
3. Determine the end behavior.
4. Sketch the intercepts and end behavior.
5. Find additional points.
6. Sketch the graph.

Example 7

Sketch the graph of the polynomial function $f(x) = 2x^4 - 8x^2$.