Functions and their Graphs

3.5 One-to-One Functions and Inverse Functions

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Definition: One-to-One Function

A function $f(x)$ is **one-to-one** if no two elements in the domain correspond to the same element in the range; that is,

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

Example 1

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function.

- $f = \{(0,0),(1,1),(1,-1)\}$
- $g = \{(-1,1),(0,0),(1,1)\}$
- $h = \{(-1,-1),(0,0),(1,1)\}$

Definition: Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, then the function is classified as a one-to-one function.

Example 2

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function. Assume that x is the independent variable and y is the dependent variable.

- $x = y^2$
- $y = x^2$
- $y = x^3$

Example 3

Determine algebraically whether the functions are one-to-one.

- $f(x) = 5x^3 - 2$
- $f(x) = |x + 1|$

Definition: Inverse Function

If f and g denote two one-to-one functions such that

$$f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f,$$

then g is the **inverse** of the function f. The function g is denoted by f^{-1} (read "f-inverse").

Domain and Range

$$\text{Domain of } f = \text{ range of } f^{-1} \textit{ and } \text{ range of } f = \text{ domain of } f^{-1}$$

$$f^{-1}(f(x)) = x \qquad \textit{and} \qquad f(f^{-1}(x)) = x.$$

Example 4

Verify that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$.

Example 5

Verify that $f^{-1}(x) = x^2$, for $x \geq 0$, is the inverse of $f(x) = \sqrt{x}$.

Finding the Inverse of a Function

- **Step 1:**
  - Let $y = f(x)$.
- **Step 2:**
  - Interchange x and y.
- **Step 3:**
  - Solve for y in terms of x.
- **Step 4:**
  - Let $y = f^{-1}(x)$.

Note:

- Verify first that a function is one-to-one prior to finding an inverse.
- State the domain restrictions on the inverse function.
- To verify that you have found the inverse, show that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.

Example 7

Find the inverse of the function $f(x) = \sqrt{x+2}$ and state the domain and range of both f and f^{-1}.

Example 8

Find the inverse of the function $f(x) = |x|$ if it exists.

Example 9

The function $f(x) = \dfrac{2}{x+3}$, $x \neq -3$, is a one-to-one function. Find its inverse.