Functions and their Graphs

3.1 Functions

September 14, 2010

Definition: Relation

A **relation** is a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *at least* one element in the second set, called the **range**.

A relation is a set of ordered pairs. For example

PERSON	BLOOD TYPE	ORDERED PAIR
Michael	A	(Michael, A)
Tania	A	(Tania, A)
Dylan	AB	(Dylan, AB)
Trevor	O	(Trevor, O)
Megan	O	(Megan, O)

The domain is $\{Michael, Tania, Dylan, Trevor, Megan\}$.

The range is $\{A, AB, O\}$.

Definition (Function)

A **function** is a correspondence between two sets where each element in the first set, called the domain, corresponds to *exactly* one element in the second set, called the **range**.

The blood-type example given is both a relation and a function.

Note: All functions are relations but not all relations are functions.

Consider the start times for some competitions on a given Saturday, at a University. The relation is not a function.

Time of Day	**Competition**
1:00 P.M.	Football
2:00 P.M.	Volleyball
7:00 P.M.	Soccer
7:00 P.M.	Basketball

Example 1

Determine whether the following relations are functions:

(a) $\{(-3,4),(2,4),(3,5),(6,4)\}$

- *No x-value is repeated. Therefore, each x-value corresponds to exactly one y-value.* This relation is a function.

(b) $\{(-3,4),(2,4),(3,5),(2,2)\}$

- *The value $x = 2$ corresponds to both $y = 2$ and $y = 4$.* This relation is not a function.

(c) Domain = Set of all items for sale in a grocery store; Range = Price

- *Each item in the grocery store corresponds to exactly one price.* This relation is a function.

Consider the equation

$$y = x^2 - 3x$$

where x can be any real number. The equation assigns to each x-value exactly one corresponding y-value. This equation represents a function.

- The variable y depends on what value of x is selected, so y is called the **dependent variable**.
- The variable x can be any number in the domain so it is called the **independent variable**.

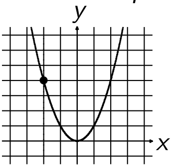
Some equations that represent functions of x:

$$y = x^2 \qquad y = |x| \qquad y = x^3$$

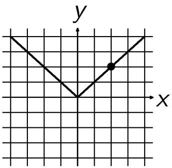
Some equations that do not represent functions of x:

$$x = y^2 \qquad x^2 + y^2 = 1 \qquad x = |y|$$

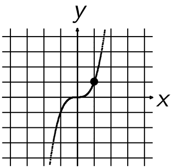
Graphs of the three functions of x



$y = x^2$

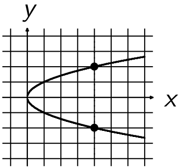


$y = |x|$

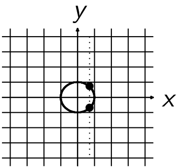


$y = x^3$

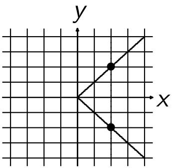
Graphs of the three functions not representing functions



$x = y^2$



$x^2 + y^2 = 1$



$x = |y|$

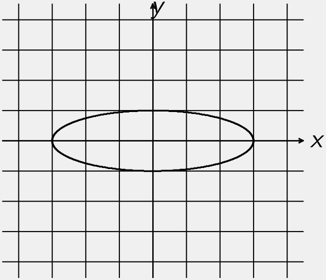
Definition (Vertical Line Test)

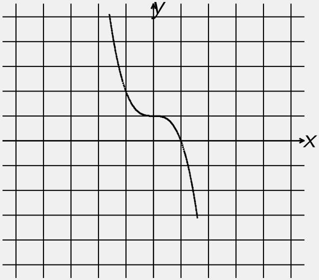
Given the graph of an equation, if any vertical line that can be drawn intersects the graph at no more than one point, the equation defines a function of x. This test is called the **vertical line test**.

Example 2

Use the vertical line test to determine whether the graphs of equations define functions of x.

(a)





Consider the equation

$$y = 2x + 5$$

which is a function because its graph is a nonvertical line (passes the vertical line test!) If we give the function a name, say, "f", then the **function notation** is:

$$f(x) = 2x + 5.$$

- The symbol $f(x)$ is read "f evaluated at x" or "f of x".
- $f(x)$ does no mean f times x.

Example 3

Given the function $f(x) = 2x^3 - 3x^2 + 6$, find $f(-1)$.

- Consider the independent variable x to be a placeholder.

$$f(\square) = 2(\square)^3 - 3(\square)^2 + 6$$

- To find $f(-1)$, substitute $x = -1$ into the function.

$$f(-1) = 2(-1)^3 - 3(-1)^2 + 6$$

- Evaluate the right side

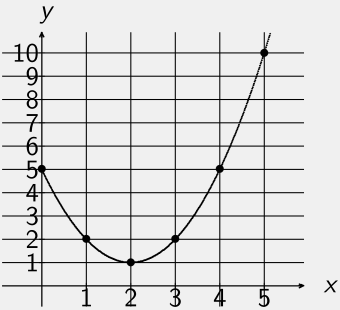
$$f(-1) = -2 - 3 + 6$$

- Simplify.

$$f(-1) = 1$$

Example 4

The graph of f is given below.



(a) Find $f(0)$. (b) Find $f(1)$. (c) Find $f(2)$.

(d) Find $4f(3)$. (e) Find x such that $f(x) = 10$.

(f) Find x such that $f(x) = 2$.

Example 5

For the given function $f(x) = x^2 - 3x$, evaluate $f(x+1)$.

Note that the function is defined with argument $\square$ and

$$f(\square) = (\square)^2 - 3(\square)$$

Example 6

For the given function $H(x) = x^2 + 2x$, evaluate

(a) $H(x+1)$
(b) $H(x) + H(1)$

Sometimes the domain of a function is stated *explicitly*. For example,

$$f(x) = |x| \qquad x < 0.$$

The **explicit domain** is the set of all negative real numbers i.e. $(-\infty, 0)$.

If the expression that defines the function is given but the domain is not stated explicitly, then the domain is implied. The **implicit domain** is the largest set of real numbers for which the function is defined and the output value $f(x)$ is a real number.

Example 10

State the domain of the given functions.

(a) $F(x) = \dfrac{3}{x^2 - 25}$ $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$

(b) $H(x) = \sqrt[4]{9 - 2x}$ $\left(-\infty, \frac{9}{2}\right)$

(c) $G(x) = \sqrt[3]{x - 1}$ $(-\infty, \infty)$