14.1 GRAPHS, PATHS AND CIRCUITS

Mathematical Concepts

April 21, 2009

Krönigsberg was an East Prussian town situated on both banks and two islands of the Prigel River.
The sections of town were connected with a series of seven bridges.

The townspeople wondered if one could walk through town and cross all seven bridges without crossing any of the bridges twice.

The question was presented to Swiss mathematician Leonhard Euler (1707-1783) who reduced the problem to one that could be represented with a series of dots and lines.

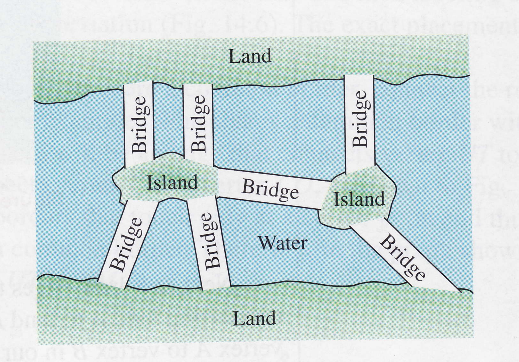


Figure 1: The Krönigsberg Bridge Problem

A **graph** is a finite set of points called **vertices** connected by line segments called **edges**. An edge that connects a vertex to itself is called a **loop**.

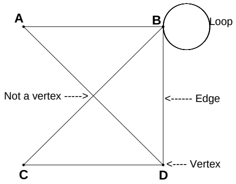


Figure 2: A Graph

The edge between two vertices will be referred to using the vertices. For example (Figure 2)

- the edge that connects the vertex A to vertex B is referred to as edge AB or as edge BA.
- the loop is referred to as edge BB or loop BB.

Not every place where two edges cross is a vertex. A dot must be present to represent a vertex.
In Figure 2, even though edges AD and BC cross, since no dot is indicated, the place where these line cross is not a vertex.

Using the definitions of vertex and edge, represent the Königsberg bridge problem with a graph.

Solution:
Note that each bridge connects two pieces of land in a manner similar to an edge connecting two vertices.

- Label each piece of land with a capital letter, as shown in Figure 3 below
- Draw edges to represent bridges.
  - Notice that there are two bridges connecting land A to land B.
  - So we need two edges to connect vertex A to vertex B in our graph.

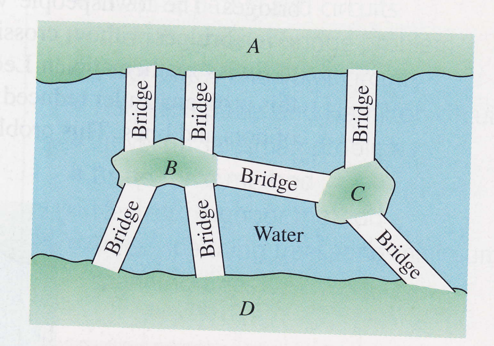


Figure 3: Labeling each piece of land with a capital letter

The land is represented with vertices and the bridges are represented by with edges.

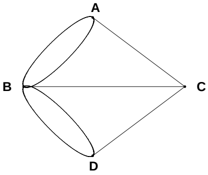


Figure 4: A Graph representing the Königsberg Bridge Problem

The Tenth United State Court of Appeals hears cases from the state shown below.

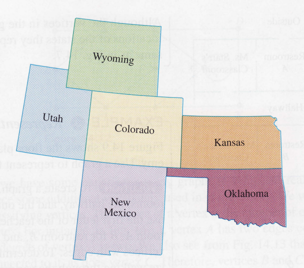


Figure 5: The Tenth U.S. Court of Appeals

Construct a graph to show the states that share a common border. States whose borders only touch at a single corner point will not be considered to share a common border.

Each vertex will represent one of the six states.

- Begin by placing the six vertices in the same relative positions as the six states. Label each vertex with the two letter state abbreviation.
- Next, if two states share a common border, connect the respective vertices with an edge.
  - For example, Utah shares a common border with Wyoming and Colorado, so there will be an edge that connects vertex UT to vertex WY and an edge that connects vertex UT to vertex CO.
  - Utah and New Mexico have borders that touch only at a corner point and thus are not considered to share a common border. Therefore, no edge connects UT to NM.

Continuing the process, we end up with the graph

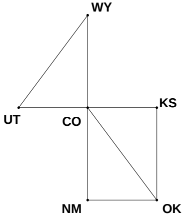


Figure 6: A Graph of The Tenth U.S. Court of Appeals

The previous graph is only one possible arrangement of vertices and edges that can show which states share a common border. Many other graphs that display the same relationship are equally valid.

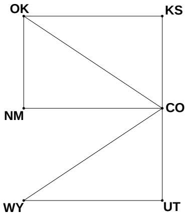
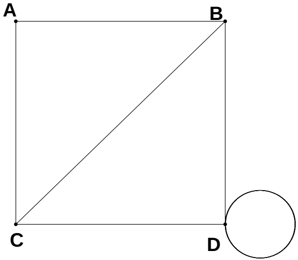


Figure 7:

Additional vocabulary used in graph theory:

The **degree** of a vertex is the number of edges that connect to that vertex. For instance, consider



- Vertex A has two edges connected to it. Therefore, the degree of vertex A is two.
- Vertices B and C each have degree three.
- Vertex D has a loop connected to it.
  - Should loop DD count as one edge or two edges when determining the degree of vertex D.
  - We will *count each end of a loop* when determining the degree of a vertex.

  Thus, vertex D will have degree four.

A vertex with an even number of edges connected to it is an **even vertex**.
A vertex with an odd number of edges connected to it is an **odd vertex**.
In the previous example, vertices A and D are even and vertices B and C are odd.

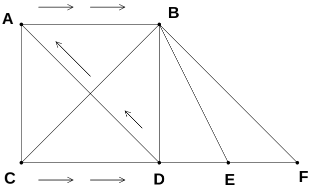
A **path** is a sequence of adjacent vertices and the edges connecting them.

Adjacent vertices means two vertices that are connected by a common edge.

- For instance, in the figure below, vertices C and D are adjacent whereas vertices A and F are not adjacent.

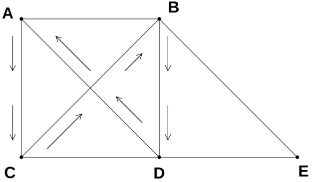
The *p*ath, given by the arrows, can be thought of as movement from vertex C to vertex D to vertex A to vertex B.

- We shall refer to this path as *path* $C,\ D,\ A,\ B$.


A
B
C
D
E
F

A **circuit** is a path that begins and ends at the same vertex.

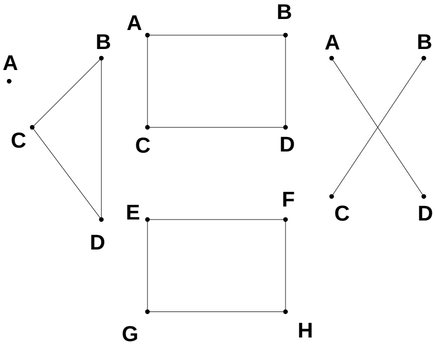
- In the figure below, the path given by $A,\ C,\ B,\ D,\ A$ forms a ciricuit.



A graph is **connected** if, for any two vertices in the graph, there is path that connects them.

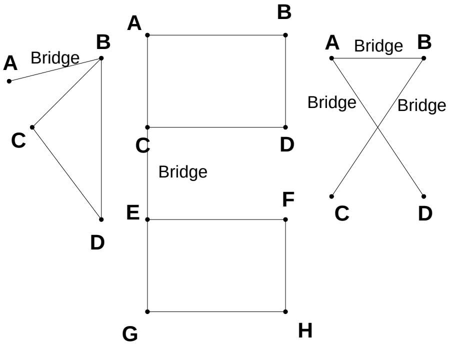
If a graph is not connected it is **disconnected**.

- Examine the graphs in the figure below. Are they connected or disconnected?



(a) (b) (c)

A **bridge** is an edge that if removed from a connected graph would create a disconnected graph.



(a) (b) (c)