12.4 EXPECTED VALUE (EXPECTATION)

Mathematical Concepts

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Expected value is often used to determine the expected results of an experiment or business venture *over the long term*.
Examples:

- Expectation is used in business to predict future profits of a new product.
- Expectation is used to determine how much each insurance policy should cost for the company to make an overall profit.
- Expectation is used to predict the expected gain or loss in games of chance such as lottery, roulette, craps and slot machines.

Expected Value

$$E = P_1 \cdot A_1 + P_2 \cdot A_2 + P_3 \cdot A_3 + \ldots + P_n \cdot A_n$$

where

- P_1 represents the probability that the first event will occur,
- A_1 represents the net amount won or lost if the first event occurs,
- P_2 represents the probability that the second event will occur,
- A_2 represents the net amount won or lost if the second event occurs,

and so on.

Example 2

a) $P(\text{guesses correctly}) = \frac{1}{5}$

$P(\text{guesses incorrectly}) = \frac{4}{5}$

$$\text{Expectation} = P_1 \cdot A_1 + P_2 \cdot A_2$$
$$= \frac{1}{5}(2) + \frac{4}{5}\left(-\frac{1}{2}\right)$$
$$= \frac{2}{5} - \frac{2}{5} = 0$$

Over the long run, she will neither gain nor lose points by guessing.

Maria is taking a multiple-choice exam in which there are five possible answers for each question. The instructions indicate that she will be awarded 2 points for each correct response, that she will lose $\frac{1}{2}$ point for each incorrect response, and that no points will be added or subtracted for answers left blank.

a) If Maria does not know the correct answer to a question, is it to her advantage or disadvantage to guess at an answer?

b) If she can eliminate one of the possible choices, is it to her advantage or disadvantage to guess at the answer?

b) $P(\text{guesses correctly}) = \frac{1}{4}$

$P(\text{guesses incorrectly}) = \frac{3}{4}$

$$E = P_1 \cdot A_1 + P_2 \cdot A_2$$
$$= \frac{1}{4}(2) + \frac{3}{4}\left(-\frac{1}{2}\right)$$
$$= \boxed{\frac{1}{8}}$$

On average she will gain $\frac{1}{8}$ point each time she guesses.

$$E = P(\text{dry}) \cdot (\text{number sold}) + P(\text{wet}) \cdot (\text{number sold})$$

$$= (0.75)(50) + (0.25)(15)$$

$$= 37.5 + 3.75 = 41.25$$

An outdoor hot dog vendor sells an average of 50 hot dogs per day in dry weather and average of 15 per day in wet weather. If the weather in this area is wet 25% of the time, determine the expected (average) number of hot dogs sold per day.

The average number of hot dogs sold per day is $\boxed{41.25}$

$P(\text{Josh wins}) = \frac{1}{100}$

$P(\text{Josh loses}) = \frac{99}{100}$

$$\text{Expectation} = P(\text{Josh wins}) \cdot (\text{amount won}) + P(\text{Josh loses}) \cdot (\text{amount lost})$$

When Josh Rosenberg attends a charity event, he is given a free ticket for the $50 door prize. A total of 100 tickets will be given out. Determine his expectation of winning the door prize.

$$\therefore E = \frac{1}{100}(50) + \frac{99}{100}(0)$$

$$= 0.50$$

Josh's expectation is $\boxed{\$\,0.50}$

$$P(\text{Josh wins}) = \frac{1}{100}$$

$$\text{net amount won} = 50 - 2 = \$48$$

$$P(\text{Josh losses}) = \frac{99}{100}$$

$$\text{amount lost} = \$2$$

When Josh Rosenberg attends a charity event, he is given the opportunity to purchase a ticket for the \$50 door prize. The cost of the ticket is \$2, and 100 tickets will be sold. Determine Josh's expectation if he purchases one ticket.

$$\text{Expectation} = P(\text{Josh wins}) \cdot (\text{amount won}) + P(\text{Josh loses}) \cdot (\text{amount lost})$$

$$= \frac{1}{100}(48) + \frac{99}{100}(-2)$$

$$= \frac{48}{100} - \frac{198}{100}$$

$$= -\frac{150}{100} = -1.50$$

Josh's expectation is $\boxed{-\$1.50}$ when he purchases one ticket.

a) If Irene wins grand prize, net gain is $\$(500-1) = \$499$

Consolation prize, net gain is $\$(100-1) = \$99$

$P(\text{grand prize}) = \frac{1}{1000}$

$P(\text{Consolation prize}) = \frac{2}{1000}$

$P(\text{not winning a prize}) = 1 - \frac{1}{1000} - \frac{2}{1000} = \frac{997}{1000}$

One thousand raffle tickets are sold for \$1 each. One grand prize of \$500 and two consolation prizes of \$100 will be awarded. The tickets are placed in a bin. The winning tickets will be selected from the bin. Assuming that the probability that any given ticket selected for the grand prize is $\frac{1}{1000}$ and the probability that any given ticket selected for a consolation prize is $\frac{2}{1000}$, determine

a) Irene Drew's expectation if she purchases one ticket.

b) Irene's expectation if she purchases five tickets.

$$E = P_1 \cdot A_1 + P_2 \cdot A_2 + P_3 \cdot A_3$$

$$= \left(\frac{1}{1000}\right)(\$499) + \left(\frac{2}{1000}\right)(\$99) + \frac{997}{1000}(-\$1)$$

$$= \frac{499}{1000} + \frac{198}{1000} - \frac{997}{1000}$$

$$= -\frac{300}{1000} = -0.30$$

Irene's expectation is $\boxed{-\$0.30}$ per ticket purchased

b) $(-\$0.3)(5) = -\boxed{\$1.50}$

The **fair price** is the amount to be paid that will result in an expected value of $0.

$$\textit{Fair price} = \textit{expected value} + \textit{cost to play}$$

a) $P(\text{wins raffle}) = \frac{1}{500}$

net amount won $= \$(400-2) = 398$

$P(\text{does not win}) = \frac{499}{500}$

amount lost $= \$2$

$$E = P(\text{wins}) \cdot (\text{amount won}) + P(\text{loses}) \cdot (\text{amount lost})$$

$$= \frac{1}{500}(\$398) + \frac{499}{500}(-\$2.00)$$

Five hundred raffle tickets are sold for $2 each. One prize of $400 is to be awarded.

a) Raul Mondesi purchases one ticket. Determine his expected value.

b) Determine the fair price of a ticket.

$$= \frac{398}{500} - \frac{998}{500}$$

$$= -\frac{600}{500} = \boxed{-\$1.20}$$

b) Fair price $= -\$1.2 + \$2.0$

$$= \boxed{\$0.80}$$