

PROBABILISTIC COMPUTATION AND STOCHASTIC DIFFERENTIAL EQUATIONS

WESLEY CALVERT

ABSTRACT. We present a foundation for the effective theory of random variables and stochastic processes. To accomplish this, we show that certain elements of the Itô stochastic calculus have a natural expression in an extension of continuous first-order (CFO) logic and are effectively true in the sense of randomized computation. In particular, the Itô stochastic integral of a computable stochastic process is also computable, and the existence of solutions to a stochastic differential equation holds in the effective category.

1. INTRODUCTION

The present paper describes an effective theory of random variables and stochastic processes. It is a common line of research in computability theory to investigate the extent to which classical results of mathematics hold true in such a way that their content is captured by algorithms. One of the most important paradigmatic results in the field is that the existence of algebraic closure is “effectively true” in the sense that there is an algorithm which will, given an algorithmic description of a field, produce an algorithmic description of its algebraic closure; in general there is no algorithm, however, which computes a function witnessing that any two effectively given algebraic closures must be isomorphic [15]. The effective truth of a result is, in some way, a minimal necessary condition for the existence of some numerical simulation of its object.

This is not the first attempt to develop an effective theory of random variables and stochastic processes. Müller [12] described some analysis on random variables based on the effective measure theory introduced by Weihrauch in [21]. Unfortunately, this approach, while promising in some applications, made even the expected value function non-computable and gave rather limited facilities for dealing with continuous-time stochastic processes [12]. These properties seem to make the environment too austere for useful modeling — we actually can, in most practically useful cases, compute expected values.

Müller chose a simple queuing system as a test case for his model of computability. The present paper chooses instead the theorem on the existence of solutions to stochastic differential equations. The importance of stochastic differential equations is well established (see [7, 23], for example), as is the literature on numerical solution of such equations [9, 14]. Moreover, while there is an analogous result on (deterministic) differential equations whose effectiveness properties are known (see [18] and [4]), the differences between the deterministic and stochastic cases are nontrivial (see the discussion in [6, 7]).

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We approach this goal by way of probabilistically computable structures in an extension of continuous first-order logic. As both probabilistically computable structures [4] and continuous first-order logic [1] are rather new notions, it is not obvious that this is the correct choice. Much of what is written in this paper could be handled exclusively in the terms of Definitions 4.1 and 4.3 and of type-2 effectiveness after the fashion of [22]. Hence, I now describe several reasons for this choice.

The first reason is historical. The original definition for “computable random variable” envisioned for this paper was Point 3 of Theorem 4.5. The fact that the approach to computability of random variables through logic was productive for the present work suggests that it may be helpful to future researchers, as well.

The motivation for this initial choice of definitions, and a reason in its own right for the logical approach, is its transparent similarity to the Monte Carlo methods which find some use in the practical numerical analysis of stochastic processes [10]. In a Monte Carlo method, the expected value of a stochastic system is simulated by taking the average of several random samples. In the present case, a single sample corresponds to the CFO structure obtained by evaluating each of the copredicates and arrays on a single element of the sample space — and the structure gives an algorithmic system of approximations to the sample path. The extended logic CFOLC incorporates all possible samples.

Another reason is that treatment of the effective treatment of random variables and stochastic processes shows us something about continuous logic. Several theorems in the later sections of this paper guarantee that a probabilistically computable CFOLC structure can be expanded to a probabilistically computable structure in which certain non-obvious features are definable. Although the satisfiability problem in CFO is not as thoroughly understood as the first-order case, it is still reasonable to take the condition that “effective structure \mathcal{M} may be expanded to an effective structure \mathcal{M}' ” as a tameness condition on the expansion.

A fourth reason for the use of logic is algebraic. In [17], Rubel suggested the unmet need for a theory of “stochastic differential algebra.” One interpretation of the results of Section 7 is that in one natural algebraization of stochastic processes, solutions to stochastic differential equations are somehow algorithmically ubiquitous. This stands in sharp contrast to the case of deterministic differential algebra, in which passing from a computable (but not decidable) copy of a field to a computable copy of its differential closure might be quite difficult.

The final reason — perhaps the most persuasive — is that many of the computations are simply more obvious in the logical language. The logic CFO on probabilistically computable structures formalizes in a mathematically natural way many of the obvious (but sometimes laborious) transfers of effectiveness when passing, e.g. to sums of functions. Capturing these features in the CFO formalism is what led to a formulation of computable random variables that trivialized Müller’s obstructions to, for instance, computing the expected value — the CFO formalism focuses our attention on the important features of the problem.

In Section 2, we will describe the logic CFOLC (continuous first-order logic with copredicates). In Section 3, we will extend the notion of probabilistic computation from [4] to cover structures in this logic. In Section 4 we define a notion of effectiveness for random variables and stochastic processes, and show that it corresponds in a natural way to definability in probabilistically computable CFOLC structures. We give some special random variables and stochastic processes as examples in Section

5. In Section 6, we show how to carry out the Itô stochastic integral in probabilistically computable structures. In Section 7, we show the effective existence of solutions to stochastic differential equations.

2. CONTINUOUS FIRST-ORDER STRUCTURES WITH COPREDICATES

We will, in keeping with the existing literature on continuous first-order logic, adopt the slightly unusual convention of using 0 as a numerical value for True (or acceptance) and 1 as a numerical value for False (or rejection). The authors of [3] chose this convention to emphasize the metric nature of their logic: the formula $d(a, b)$, which plays a role analogous to that of equality in first-order logic, is true if its value is 0.

Continuous first-order logic is an extension of Łukasiewicz propositional logic, which builds on work of Keisler and Henson (see [1] for a more detailed history). Formulas in continuous first-order logic (CFO) take truth values in the closed unit interval, and every continuous function on a finite Cartesian power of the unit interval is a logical connective. The following definition is from [2].

Definition 2.1. A *continuous signature* is an object of the form $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{G}, n)$ where

- (1) \mathcal{R} and \mathcal{F} are disjoint and \mathcal{R} is nonempty, and
- (2) n is a function associating to each member of $\mathcal{R} \cup \mathcal{F}$ its arity
- (3) \mathcal{G} has the form $\{\delta_{s,i} : (0, 1] \rightarrow (0, 1] : s \in \mathcal{R} \cup \mathcal{F} \text{ and } i < n_s\}$

Members of \mathcal{R} are called *predicate symbols*, and members of \mathcal{F} *function symbols*.

We will work in an extension of continuous first-order logic, which we call continuous first-order logic with copredicates (CFOLC).

Definition 2.2. A *CFOLC signature* is an object of the form $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C}, \mathcal{K}, \mathcal{G}, n)$ where

- (1) $\mathcal{R}, \mathcal{F}, \mathcal{C}$, and \mathcal{K} are disjoint and \mathcal{R} is nonempty, and
- (2) n is a function associating to each member of $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C} \cup \mathcal{K}$ its arity
- (3) \mathcal{G} has the form $\{\delta_{s,i} : (0, 1] \rightarrow (0, 1] : s \in \mathcal{R} \cup \mathcal{F} \cup \mathcal{C} \cup \mathcal{K} \text{ and } i < n_s\}$

We call the elements of \mathcal{C} copredicates, and the elements of \mathcal{K} arrays. Let I denote the closed unit interval, $I(K) = I \cap K$, and $I^c = I(K)$ where K is the set of computable reals. We also make free use of the effective equivalence between 2^ω and I . We now define the class of structures.

Definition 2.3. Let $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C}, \mathcal{K}, \mathcal{G}, n)$ be a continuous signature. A *CFOLC \mathcal{L} -pre-structure* is an ordered pair $\mathfrak{M} = (M, \rho)$, where M is a non-empty set, and ρ is a function on $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}, \mathcal{K}$ such that

- (1) To each function symbol f , the function ρ assigns a mapping $f^{\mathfrak{M}} : M^{n(f)} \rightarrow M$
- (2) To each predicate symbol P , the function ρ assigns a mapping $P^{\mathfrak{M}} : M^{n(P)} \rightarrow I$.
- (3) To each copredicate symbol C , the function ρ assigns a mapping $C^{\mathfrak{M}} : I^c \rightarrow M^{n(C)}$.
- (4) To each array symbol K , the function ρ assigns a mapping $K^{\mathfrak{M}} : I^c \rightarrow \mathcal{F}$, where for each x , we have $n_{K(x)} = n_K$.
- (5) The function ρ assigns d to a pseudo-metric $d^{\mathfrak{M}} : M \times M \rightarrow I$.

(6) For each $f \in \mathcal{F}$ for each $i < n_f$, and for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e [d^{\mathfrak{M}}(c, e) < \delta_{f,i}(\epsilon) \Rightarrow d^{\mathfrak{M}}(f^{\mathfrak{M}}(\bar{a}, c, \bar{b}), f^{\mathfrak{M}}(\bar{a}, e, \bar{b})) \leq \epsilon]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_f - 1$.

(7) For each $P \in \mathcal{R}$ for each $i < n_P$, and for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e [d^{\mathfrak{M}}(c, e) < \delta_{P,i}(\epsilon) \Rightarrow |P^{\mathfrak{M}}(\bar{a}, c, \bar{b}) - P^{\mathfrak{M}}(\bar{a}, e, \bar{b})| \leq \epsilon]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_P - 1$.

(8) For each $C \in \mathcal{C}$ for each $i < n_C$, and for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e [d^{\mathfrak{M}}(c, e) < \delta_{C,i}(\epsilon) \Rightarrow |C^{\mathfrak{M}}(\bar{a}, c, \bar{b}) - C^{\mathfrak{M}}(\bar{a}, e, \bar{b})| \leq \epsilon]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_C - 1$.

(9) For each $K \in \mathcal{K}$, for each $i < n_K$, and for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e \left[d^{\mathfrak{M}}(c, e) < \delta_{K,i}(\epsilon) \Rightarrow \int_0^1 [K(\bar{a}, c, \bar{b})(x) - K(\bar{a}, e, \bar{b})(x)]^2 dx \leq \epsilon \right]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_K - 1$.

If we specialize to the case $\mathcal{C} = \mathcal{K} = \emptyset$, then we have exactly the corresponding definitions for CFO.

Definition 2.4. A *CFOLC \mathcal{L} -structure* is a CFOLC \mathcal{L} -pre-structure such that ρ assigns to d a metric.

Since we are concerned here with countable structures, we will not use the stronger notion common in the CFO literature, which requires that d be assigned to a *complete* metric. However, it is possible, given a continuous weak structure (even a pre-structure), to pass to a completion [2].

We now define the syntax and semantics of this logic. Following the reasoning in [1] and [2], it suffices to describe the syntax and semantics for a logic which admits approximation of every formula (those familiar with quantum computation will find this approach familiar [13]).

Definition 2.5. The class of CFOLC terms is exhaustively defined by the following induction:

- (1) Every variable is a term.
- (2) Every copredicate symbol is a term.
- (3) If t_1, \dots, t_n are terms and f is a function symbol, then $f(t_1, \dots, t_n)$ is a term.
- (4) If t_1, \dots, t_n are terms and K is an array symbol, then $K(t_1, \dots, t_n)$ is a term.

Definition 2.6. The atomic formulas are the expressions of the form $P(t_1, \dots, t_n)$ in which P is an n -ary predicate symbol and t_1, \dots, t_n are terms, or of the form $d(t_1, t_2)$, where t_1, t_2 are terms. The formulas of CFOLC are defined inductively, exactly as in continuous first-order logic.

Definition 2.7. Let \mathcal{M} be a CFOLC structure in a signature that includes a copredicate X . Then \tilde{X} is the unique continuous extension of X mapping $[0, 1]$ to the Cauchy completion of M .

Definition 2.8. Let \mathcal{M} be a CFOLC structure in a signature that includes an array K . Then \tilde{K} is the unique continuous extension of K mapping $[0, 1]$ to the Cauchy completion of the the set $\mathcal{F}^{\mathcal{M}}$.

Recall that the expected value of a random variable $\tilde{X} : I \rightarrow S$ is defined by $\int_I \tilde{X}$. To speak of the expected value of expressions involving n -ary arrays, we treat them as functions of $n + 1$ variables: one from I , and n from the structure. If P is a predicate, then, and $K(x) = K(\theta, x)$, where θ is the variable from I , is an array, we say that the expected value of $P(K(x))$ is given by $\int_I P(K(\theta, x))d\theta$.

We now define satisfaction of a formula.

Definition 2.9. Let V denote the set of variables, and let $\sigma : V \rightarrow M$. Let φ be a formula.

- (1) The *interpretation under σ* of a term t (written $t^{\mathfrak{M}, \sigma}$) is defined by replacing each variable x in t by $\sigma(x)$.
- (2) If P is an n -ary predicate symbol and

$$t_1(X_1, \dots, X_k, K_1, \dots, K_j), \dots, t_n(X_1, \dots, X_k, K_1, \dots, K_j)$$

are terms involving copredicates X_1, \dots, X_k and arrays K_1, \dots, K_j , then we define $P^{\mathfrak{M}}(\bar{t}^{\mathfrak{M}, \sigma})$ to be the expected value of

$$P^{\mathfrak{M}}(t_1(\tilde{X}_1, \dots, \tilde{X}_k, \tilde{K}_1, \dots, \tilde{K}_j), \dots, t_n(\tilde{X}_1, \dots, \tilde{X}_k, \tilde{K}_1, \dots, \tilde{K}_j).$$

- (3) Let φ be a formula. We then define the *value of φ in \mathfrak{M} under σ* (written $\mathfrak{M}(\varphi, \sigma)$) as follows:
 - (a) $\mathfrak{M}(P(\bar{t}), \sigma) := P^{\mathfrak{M}}(\overline{t^{\mathfrak{M}, \sigma}})$
 - (b) $\mathfrak{M}(\alpha \dot{-} \beta, \sigma) := \max(\mathfrak{M}(\alpha, \sigma) - \mathfrak{M}(\beta, \sigma), 0)$
 - (c) $\mathfrak{M}(\neg\alpha, \sigma) := 1 - \mathfrak{M}(\alpha, \sigma)$
 - (d) $\mathfrak{M}(\frac{1}{2}\alpha, \sigma) := \frac{1}{2}\mathfrak{M}(\alpha, \sigma)$
 - (e) $\mathfrak{M}(\sup_x \alpha, \sigma) := \sup_{a \in M} \mathfrak{M}(\alpha, \sigma_x^a)$, where σ_x^a is equal to σ except that $\sigma_x^a(x) = a$.
- (4) We write $(\mathfrak{M}, \sigma) \models \varphi$ exactly when $\mathfrak{M}(\varphi, \sigma) = 0$.

Of course, if φ has no free variables, then the value of $\mathfrak{M}(\varphi, \sigma)$ is independent of σ .

3. PROBABILISTICALLY COMPUTABLE CFOLC STRUCTURES

In [4], the present author described a model of computation for countable CFO weak structures (i.e. CFO structures with a metric which need not be complete). We extend the definitions given there to the context of CFOLC.

If M is a Turing machine, we write $M^x(n)$ for the result of applying M to input n with oracle x . Excepting the polarity change to match the conventions above, the following definition is standard; it may be found, for instance, in [19].

Definition 3.1. Let 2^ω be the set of infinite binary sequences, with the usual Lebesgue probability measure μ .

- (1) A *probabilistic Turing machine* is a Turing machine equipped with an oracle for an element of 2^ω , called the *random bits*, with output in $\{0, 1\}$ (note that this is merely an oracle machine; the nomenclature emphasizes the use we will make of the oracle space as sample space).

- (2) We say that a probabilistic Turing machine M *accepts* n with probability p if and only if $\mu\{x \in 2^\omega : M^x(n) \downarrow = 0\} = p$.
- (3) We say that a probabilistic Turing machine M *rejects* n with probability p if and only if $\mu\{x \in 2^\omega : M^x(n) \downarrow = 1\} = p$.

Definition 3.2. Let \mathcal{L} be a computable CFOLC signature. Let \mathfrak{M} be a CFOLC \mathcal{L} -structure. Let $\mathcal{L}(\mathfrak{M})$ be the expansion of \mathcal{L} by a constant c_m for each $m \in M$ (i.e. a unary predicate $c_m \in \mathcal{R}$ where $c_m^{\mathfrak{M}}(x) := d(x, m)$). Then the *CFOLC atomic diagram* of \mathfrak{M} , written $D(\mathfrak{M})$ is the set of all pairs (φ, p) , where φ is a quantifier-free (i.e. sup- and inf-free) sentence in $\mathcal{L}(\mathfrak{M})$ and $\mathfrak{M}(\varphi, \sigma) = p$. The CFOLC elementary diagram $D^*(\mathfrak{M})$ is the same, except that φ is not required to be quantifier-free.

Note that the definition is independent of σ , since a sentence has no free variables.

Definition 3.3. We say that a CFOLC pre-structure \mathfrak{M} is *probabilistically computable* (respectively, *probabilistically decidable*) if and only if there is some probabilistic Turing machine T such that, for every pair $(\varphi, p) \in D(\mathfrak{M})$ (respectively, $D^*(\mathfrak{M})$) the machine T accepts φ with probability p .

4. PROBABILISTICALLY COMPUTABLE RANDOM VARIABLES AND STOCHASTIC PROCESSES

Stochasticity is the reason for including copredicates and arrays in the logic. In economic terms, stochasticity is an *externality* — a feature not accounted for in the structure itself, represented as a dependence on a variable from the sample space (in general, an arbitrary measure space of total measure 1; here, we make use of the fact that in most practically useful cases, I suffices). Copredicates will be used to represent random variables (maps from a sample space into the structure), and arrays will be used to represent stochastic processes (functions of two variables: one from the structure, and one from a sample space).

In the present section, we will define the notion of probabilistic computability on both random variables and stochastic processes. The definition in each case starts with the rather obvious one from the perspective of type-2 effectiveness and the related school of computable analysis. In each case, the definition will not be obviously connected to randomized computation of the sort used in the previous section. The justification for the names comes, rather, from Theorems 4.5 and 4.7, which also explain the usefulness of copredicates and arrays in probabilistically computable structures. We will have occasion to mention the *distribution* of a random variable \tilde{X} . The distribution of \tilde{X} is the function $F_{\tilde{X}}(a) = Pr(\tilde{X} < a)$.

Definition 4.1. A random variable $\tilde{X} : 2^\omega \rightarrow I$ is said to be *computable* if and only if there is an oracle Turing machine Φ such that Φ^θ computes the real number $\tilde{X}(\theta)$.

We let the random variable take values in I to guarantee its uniform continuity. However, since there is an effective bijection of $(0, 1)$ with \mathbb{R} , we may freely interpret a random variable with values in I as one with arbitrary range in \mathbb{R} .

However, the exact function \tilde{X} is not the elementary concept for probability. Rather, as Loève explains in what he calls *the essential feature of probability theory*, “A property of a family of functions on a measure space is pr[obability]-theoretical if, and only if, the property remains the same when the family is replaced by

any other family with the same distribution” [11]. Computability, in the sense of Definition 4.1, is not invariant up to equal distribution.

Example 4.2. Let σ be a permutation of ω . Then let $\tilde{X}(\theta)(n) = \text{XOR}(\theta(n), \sigma(n))$. Since \tilde{X} is measure-preserving, the distribution of \tilde{X} is the same as that of the identity function. However, \tilde{X} is of the same degree as σ .

Thus, computability of a random variable is not invariant in the same sense that computability of a first-order structure is not invariant under isomorphism. We address this by noting that the distribution determines the random variable, not uniquely, but up to a strong equivalence relation. The distribution of \tilde{X} determines $\text{Pr}(\tilde{X} \in B)$ for any Borel set B , and in particular it determines the value of $E(\tilde{X})$ — indeed, of any Borel function of \tilde{X} [8].

Definition 4.3. A random variable $\tilde{X}(\theta) : 2^\omega \rightarrow I$ is said to be *probabilistically computable* if and only if it has the same distribution as a computable random variable.

We will often follow the usual convention of computable structure theory in speaking of an *index for \tilde{X}* when \tilde{X} is a probabilistically computable random variable; we mean here, of course, an index for some computable random variable with the same distribution. We will also mean by a *copy of \tilde{X}* another random variable with the same distribution as \tilde{X} .

Proposition 4.4. *There is a Turing machine \mathcal{E} which, given an index for a probabilistically computable random variable \tilde{X} , will produce an index for the (necessarily computable) real number $E(\tilde{X}) := \int_{2^\omega} \tilde{X} d\mu$, where μ is the uniform measure on 2^ω .*

Proof. By Theorem 3.2.11 of [22] (in essence, all Type-2 computable functions are continuous), the function \tilde{X} must be continuous, and so the Lemma holds by Theorem 6.4.1 of [22] (stating that there is a Turing machine that will pass from a computable index for a continuous Type-2 computable function to an index for its integral). \square

For a random variable \tilde{X} , we will write X for $\tilde{X} \upharpoonright_{I^c}$. Note that if \tilde{X} is computable, then $X : I^c \rightarrow I^c$.

Theorem 4.5. *Let \tilde{X} be a random variable. The following are equivalent:*

- (1) \tilde{X} is probabilistically computable.
- (2) If \mathcal{M} is a probabilistically computable CFOLC structure whose universe contains I^c , then the expansion of \mathcal{M} by a copredicate interpreted as some copy of X remains probabilistically computable.
- (3) There is a probabilistically computable CFOLC structure whose universe contains I^c in which a copy of X is quantifier-free definable.
- (4) The distribution for \tilde{X} is computable, in the sense that there is a Turing machine which will, given a rational number a , compute the real number $\text{Pr}(\tilde{X} < a)$.

Proof. To show 1 \rightarrow 2, let \mathcal{M} be a probabilistically computable CFOLC structure whose universe contains I^c , and suppose $\varphi(\bar{t}, X)$ is a CFOLC formula of $\mathcal{M}(X)$, where \bar{t} are CFOLC terms of \mathcal{M} . Without loss of generality, we may suppose that we have a computable copy of \tilde{X} . As in Lemma 4.4, we can effectively pass to a machine T which computes the digits of $\mathcal{E}(\varphi(\bar{t}, X))$. Now to accept $\varphi(\bar{t}, X)$ with

probability equal to $\mathcal{E}(\varphi(\bar{t}, X))$, we accept on any string which is known to be less than the value of the real computed by T , and reject on any string known to be greater than the value of that real.

The implication $2 \rightarrow 3$ is trivial. We now show that $3 \rightarrow 4 \rightarrow 1$. Suppose that (I^c, X) is probabilistically computable, with X as a copredicate.

Let $\varphi(a, e, x)$ be the predicate which takes value 1 for $x < a$, value 0 for $x \geq a + e$, and value $-\frac{1}{e}x + 1 + \frac{a}{e}$ on $[a, a + e]$. Now a uniform family of approximations to the distribution for X is given by $F_{X,u}^e(a) = \int_{I(\mathbb{Q})} \varphi(a, e, x)$. As in Lemma 4.4, we can uniformly pass from an index for (I^c, X) and a code for a to an index for $F_{X,u}^e(a)$. Note that for $e_1 > e_2$ we have $F_{X,u}^{e_1} < F_{X,u}^{e_2}$, and that $\lim_{e \rightarrow \infty} F_{X,u}^e(a)$ is the distribution for X . Similarly, we can get a sequence of approximations $F_{X,\ell}^e$, strictly monotonically increasing in e , which also converge to the distribution for X . This establishes 4.

In the following construction toward 1, we will have sets of ‘‘committed’’ intervals: all elements in intervals in $C_{>,a}$ will be mapped to elements larger than a , and symmetrically for $<$. We will also speak of sets $K_{*,a,s}$ which record elements to be added to $C_{*,a}$.

Now let $\Gamma_0 = C_{>,a,0} = C_{<,a,0} = \emptyset$ for all a . At stage s , for each $a \in I(\mathbb{Q})$ with code less than or equal to s , we look for the pair $\langle k_{1,a,s}, n_{1,a,s} \rangle \leq s$ such that $\frac{k_{1,a,s}}{2^{n_{1,a,s}}}$ is the least such that $F_{X,u}^s(a) < \frac{k_{1,a,s}}{2^{n_{1,a,s}}}$, and the pair $\langle k_{2,a,s}, n_{2,a,s} \rangle \leq s$ such that $\frac{k_{2,a,s}}{2^{n_{2,a,s}}}$ is the greatest such that $F_{X,u}^s(a) > \frac{k_{2,a,s}}{2^{n_{2,a,s}}}$. We now choose finite disjoint sets $K_{>,a,s}$ and $K_{<,a,s}$, each disjoint from both $C_{>,a,s}$ and $C_{<,a,s}$, satisfying the following properties:

- (1) If $k_{1,a,s} \geq k_{1,a,t}$ for some $t < s$, then $K_{>,a,s} = \emptyset$. Otherwise,
 - (a) $\sum_{x \in K_{>,a,s} \cup C_{>,a,s}} lh(x) = 1 - \frac{k_{1,a,s}}{2^{n_{1,a,s}}}$,
 - (b) For all $b < a$ and all $x \in K_{>,a,s}$ we do not have $x \in C_{<,b,s}$, and
- (2) If $k_{2,a,s} \leq k_{2,a,t}$ for some $t < s$, then $K_{<,a,s} = \emptyset$. Otherwise,
 - (a) $\sum_{x \in K_{<,a,s} \cup C_{<,a,s}} lh(x) = \frac{k_{2,a,s}}{2^{n_{2,a,s}}}$,
 - (b) For all $b > a$ and all $x \in K_{<,a,s}$ we do not have $x \in C_{>,b,s}$.

We now set $C_{*,a,s+1} = C_{*,a,s} \cup K_{*,a,s}$ for $* \in \{<, >\}$. We also extend Γ as follows. Let \mathcal{D} be the Boolean algebra generated by the intervals (specified by initial segments) comprising $\bigcup_a C_{<,a,s+1} \cup C_{>,a,s+1}$, and let D be the element of \mathcal{D} with minimal code. Since there are only finitely many a involved, we can find an interval $R \subseteq [0, 1]$, with rational endpoints, such that

- (1) For all a , if D is contained in a member of $C_{<,a,s+1}$ then all members of R are less than or equal to a ,
- (2) For all a , if D is contained in a member of $C_{>,a,s+1}$, then all members of R are greater than or equal to a , and
- (3) R is disjoint from all intervals in the range of Γ_s .

We then define $\Gamma_{s+1} = \Gamma_s \cup \{(D, R)\}$, and set $\Gamma = \bigcup_s \Gamma_s$. Now let Φ be such that for any $D \in \text{dom}(\Gamma)$ we have $\Phi^D = \Gamma(D)$. This completes the proof. \square

We extend the discussion at this point to include stochastic processes.

Definition 4.6.

- (1) A stochastic process $\tilde{\xi}(t, \theta)$ is said to be *computable* if and only if there is an oracle Turing machine Φ such that $\Phi^\theta(s, t) = \tilde{\xi}(t, \theta)$ for all t, θ , where t ranges over all codes for elements of I , in the sense that $\Phi^\theta(s, t)$ computes the real number $\tilde{\xi}(t, \theta)$. (Here the variable s tracks the time in the computation of the digits of the real number.)
- (2) A stochastic process $\tilde{\xi}(t, \theta)$ is said to be *probabilistically computable* if and only if there is some computable stochastic process $\tilde{\zeta}(t, \theta)$ such that for each t , the random variables $\tilde{\xi}(t, \theta)$ and $\tilde{\zeta}(t, \theta)$ have the same distribution.

As before, if $\tilde{\xi}$ is a stochastic process, we write ξ for the largest restriction of $\tilde{\xi}$ to \mathbb{Q} which takes values in \mathcal{F} . Our usage of the terminology for indices and copies is analogous to the case of random variables.

Theorem 4.7. *Let $\tilde{\xi}$ be a stochastic process. Then the following are equivalent:*

- (1) $\tilde{\xi}$ is probabilistically computable.
- (2) If \mathcal{M} is a probabilistically computable CFOLC structure whose universe includes I^c , then the expansion of \mathcal{M} by an array interpreted as some copy of ξ remains probabilistically computable.
- (3) There is a probabilistically computable CFOLC structure in which a copy of ξ is quantifier-free definable.

Proof. For $1 \rightarrow 2$, assume without loss that we have a computable copy of $\tilde{\xi}$, let \mathcal{M} be a probabilistically computable CFOLC structure, and let $\varphi(\bar{t})$ be a CFOLC formula of (\mathcal{M}, ξ) . As in Theorem 4.5, we can effectively pass to the index of a machine T which gives the expected value of $\varphi(\bar{t})$ as a computable real (Corollary 6.4.2 of [22] allows integration over many variables, as may happen if we have more than one application of ξ in $\varphi(\bar{t})$). Also as in Theorem 4.5, we arrange to accept the formula according to the computation of that machine.

The implication $2 \rightarrow 3$ is obvious. For $3 \rightarrow 1$, we suppose that (I^c, ξ) is probabilistically computable. Following the proof of Theorem 4.5, we have a uniform procedure for passing from a value \bar{t} to an index for the probabilistically computable random variable $\tilde{\xi}(\bar{t})$. By the *s-m-n* Theorem (see [20]), we can then pass to an index for the probabilistically computable stochastic process $\tilde{\xi}$. \square

5. CONSTRUCTION OF SPECIAL RANDOM VARIABLES AND STOCHASTIC PROCESSES

In view of point 4 of Theorem 4.5, most standard random variables are probabilistically computable. Because the general proof of that theorem — and especially the most relevant point — is not intuitively transparent, though, the following example is given by a different method.

Lemma 5.1. *There is a uniform collection $(\eta_i)_{i \in \omega}$ of independent computable standard normal random variables.*

Proof. We construct the collection $(\eta_i)_{i \in \omega}$ on \mathbb{R} , and note that we can replace them by functions compressed into I . We recall from Section 11.3 of [16] that if U_1 and U_2 are independent uniform random variables on I , then

$$X_1 = (-2 \ln U_1)^{\frac{1}{2}} \cos(2\pi U_2)$$

and

$$X_2 = (-2 \ln U_1)^{\frac{1}{2}} \sin(2\pi U_2)$$

are independent normal random variables with mean 0 and standard deviation 1. Let $[x]_n$ denote the truncation of x to n places, and let $\{f\}_n$ denote the n th Taylor approximation to f . Now we write $X_{i,n}$ for the result of replacing, in the formula for X_i , each instance of U_j by $[U_j]_n$ and each instance of a non-polynomial function f by $\{f\}_n$. Since the Taylor series converge with $O((n+1)!)$, the limit $X_i = \lim_{n \in \omega} X_{i,n}$ is a computable real number. Since (Theorem 6.2.2 of [22]) the operation carrying a sequence of computable functions to the limit of the sequence (again a computable function) is computable, so that we can uniformly pass from indices for U_1 and U_2 to indices for X_1 and X_2 . Since one can produce an infinite family of independent uniform probabilistically computable random variables, the result follows. \square

The computability of important stochastic processes is not as obvious from Theorem 4.7. The following example is of fundamental importance for the sequel.

Proposition 5.2. *There is a probabilistically computable CFOLC structure in which a Wiener process is quantifier-free definable.*

Proof. The construction of the process itself will follow the exposition in [5]. From a logical standpoint, we start with a probabilistically decidable structure \mathcal{M} in which an infinite set of independent normal random variables is definable, and will construct a uniformly effective sequence of approximations to the Wiener process. We will then show that \mathcal{M} may be expanded by an array for the Wiener process without losing computability.

We first define a set of component wavelets, and then express the Wiener process in terms of these. For each $k \in \omega$, we define the k th Schauder function s_k by first finding the unique n such that $2^n \leq k < 2^{n+1}$, and then letting

$$s_k(t) := \begin{cases} 2^{\frac{n}{2}}(t+1) - k2^{-\frac{n}{2}} & \text{if } \frac{k-2^n}{2^n} \leq t \leq \frac{k-2^n+\frac{1}{2}}{2^n} \\ -2^{\frac{n}{2}}(t+1) + (k+1)2^{-\frac{n}{2}} & \text{if } \frac{k-2^n+\frac{1}{2}}{2^n} < t \leq \frac{k-2^n+1}{2^n} \\ 0 & \text{otherwise} \end{cases} .$$

Thus, the graph of s_k looks like a triangular tent of height $2^{-\frac{n+2}{2}}$ over a single interval, and is identically zero elsewhere. Note that for any \hat{k} there is a CFOLC formula $\check{s}_{\hat{k}}(\kappa, x)$ such that $\check{s}_{\hat{k}}(2^{-k}, x) = s_k(x)$ for all $k \leq \hat{k}$, since for each natural k the function s_k is uniformly continuous, and there is a uniformly continuous extension to the entire unit square.

Now let $(\eta_i)_{i \in \omega}$ be a sequence of independent normal random variables definable in \mathcal{M} . From [5], we see that

$$W(t, \theta) := \sum_{k=0}^{\infty} \eta_k(\theta) s_k(t)$$

is a Wiener process. Denote by $W_k(t, \theta)$ the partial sum of the first k terms. Now we can pass uniformly from an index for the family $(\eta_i)_{i \in \omega}$ to an index for the family of functions $(W_k(t, \theta))_{k \in \omega}$, which are all (uniformly) probabilistically computable by Theorem 4.7. Now we pass, as before, to the limit, $W(t, \theta) = \lim_{k \rightarrow \infty} W_k(t, \theta)$ using Theorem 6.2.2 of [22]. We now have a probabilistically computable stochastic process, and, by Theorem 4.7, a probabilistically computable CFOLC structure in which it is definable without quantifiers. \square

6. PROBABILISTICALLY COMPUTABLE ITÔ CALCULUS

Theorem 6.1. *If w is a Wiener process and $\mathcal{M} = (I^c, \xi, w)$ is probabilistically computable, then there is a probabilistically computable expansion of \mathcal{M} in which $\int_0^1 \xi dw$ is definable without quantifiers, provided only that the integral exists.*

Proof. Assume the integral exists. Then $\mathcal{I} = \int_0^t \xi dw$ is a function on 2^ω , defined by analogy with the Riemann-Stieljes integral (see [6, 7] for full details). In particular, if

$$\Gamma = \{0 = \gamma_0 < \gamma_1 < \dots < \gamma_m\} = 1$$

is a partition of the unit interval, then we write $|\Gamma|$ for the maximum distance $\gamma_{i+1} - \gamma_i$, and we define

$$S_\Gamma = \sum_{i=1}^m \xi(\gamma_{i-1})[w(\gamma_i) - w(\gamma_{i-1})],$$

and we define

$$\int_0^1 \xi dw = \lim_{|\Gamma| \rightarrow 0} S_\Gamma.$$

Note that S_Γ is CFO definable in \mathcal{M} without quantifiers using only the elements of Γ as parameters. The use of $\xi(\gamma_{i-1})$ in the definition of S_Γ is the essence of the difference between the definition of the Itô integral and the Riemann-Stieljes integral (the latter would use an arbitrary $\alpha_i \in [\gamma_{i-1}, \gamma_i]$ in place of γ_{i-1}). Since the functions involved may not be of bounded variation, the standard Riemann-Stieljes integral is, in general, undefined here. Indeed, the choice of α_i indexes an infinite number of definitions of the integral, with sometimes wildly differing calculus (e.g. the standard chain rule holds if $\alpha_i = \frac{\gamma_{i-1} + \gamma_i}{2}$ as in the Stratonovich stochastic calculus, but the choice of $\alpha_i = \gamma_{i-1}$ characteristic of the Itô approach allows the integral, as a function of its upper limit, to be a martingale). The present proof would apply equally to any uniform choice of α_i .

Now we take a sequence $(\Gamma_i)_{i \in \omega}$ with $\lim_{i \rightarrow \infty} |\Gamma_i| = 0$. By Theorem 6.2.2 of [22], we can effectively pass to the limit $\lim_{i \rightarrow \infty} S_{\Gamma_i}$, which is now a computable stochastic process. Using Theorem 4.7, the result follows. \square

The proof applies, *mutatis mutandis*, to integration over any subinterval of I with computable endpoints.

7. SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS

The classical theorem on existence of solutions to deterministic differential equations is equivalent to the formal system WKL_0 , as is discussed at length in [18]. In particular, it is not effectively true in the classical sense. However, probabilistically computable CFO structures implement effectiveness at the level of ACA_0 , so that the existence theorem for deterministic differential equations is true in this category (see [4]). The stochastic analogue to the classical result on existence and uniqueness of solutions to ordinary differential equations is this (see [7]):

Theorem 7.1. *Let w be a Wiener process, and suppose the following:*

- (1) $f(t, x)$ and $g(t, x)$ are measurable,
- (2) There is a constant K such that for all $t \in I; x, y \in \mathbb{R}$, we have
 - (a) $|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq K|x - y|$, and

- (b) $|f(t, x)|^2 + |g(t, x)|^2 \leq K^2(1 + |x|^2)$, and
 (3) ζ_0 is independent of $w(t)$ for $t > 0$, with $E(\zeta_0^2) < \infty$.

Then there is a stochastic process $\zeta(t)$ with the following properties:

- (1) $\zeta(t)$ satisfies the equation

$$d\zeta(t) = f(t, \zeta(t))dt + g(t, \zeta(t))dw,$$

such that $\zeta(0) = \zeta_0$,

- (2) $\zeta(0) = \zeta_0$,
 (3) $\zeta(t)$ is continuous with probability 1, and
 (4) $\sup_I E(\zeta^2(t)) < \infty$.

We now prove an effective version of this theorem from the perspective of probabilistically computable stochastic processes.

Theorem 7.2 (Effective Existence). *If \mathcal{M} is a probabilistically computable structure and φ is a stochastic differential equation of the form*

$$d\zeta = f(t, \zeta(t))dt + g(t, \zeta(t))dw, \quad \zeta(0) = \zeta_0,$$

where functions f and g and a Wiener process w are quantifier-free definable in \mathcal{M} and the conditions of Theorem 7.1 are satisfied, then there is a probabilistically computable expansion \mathcal{M}' of \mathcal{M} such that φ has a solution which is definable in \mathcal{M}' .

Proof. The standard existence proof (see [7, 6] for classical treatments), with some added attention to effectiveness, suffices. We construct a sequence of successive approximations as follows:

$$\begin{aligned} \zeta_0(t) &= \zeta_0 \\ \zeta_n(t) &= \zeta_0 + \int_0^t f(s, \zeta_{n-1}(s))ds + \int_0^t g(s, \zeta_{n-1}(s))dw(s) \end{aligned}$$

Assuming, for a moment, that the limit $\lim_{n \rightarrow \infty} \zeta_n$ exists, it is not difficult to see that it is a solution to φ , since passing to the limit in the definition of ζ_n gives us the integral form of φ .

We have observed many times that we may effectively pass from indices for f and ζ_{n-1} to an index for the probabilistically computable random variable

$$\int_0^t f(s, \zeta_{n-1}(s))ds.$$

By Theorem 6.1, we may do the same for

$$\int_0^t g(s, \zeta_{n-1}(s))dw(s).$$

Inductively, then, we have an effective sequence of indices for the probabilistically computable stochastic processes $(\zeta_n)_{n \in \omega}$. Thus, we obtain an index for the limit ζ (as a computable stochastic process), using Theorem 6.2.2 of [22], and the definability follows by Theorem 4.7. \square

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DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY HALL 6C, MURRAY STATE UNIVERSITY,
MURRAY, KENTUCKY 42071

E-mail address: wesley.calvert@murraystate.edu