

Index sets for classes of high rank structures

W. Calvert, E. Fokina, S. S. Goncharov, J. F. Knight,
O. Kudinov, A. S. Morozov, V. Puzarenko *

April 27, 2007

Abstract

This paper calculates, in a precise way, the complexity of the index sets for three classes of computable structures: the class $K_{\omega_1^{CK}}$ of structures of Scott rank ω_1^{CK} , the class $K_{\omega_1^{CK}+1}$ of structures of Scott rank $\omega_1^{CK}+1$, and the class K of all structures of non-computable Scott rank. We show that $I(K)$ is m -complete Σ_1^1 , $I(K_{\omega_1^{CK}})$ is m -complete Π_2^0 relative to Kleene's \mathcal{O} , and $I(K_{\omega_1^{CK}+1})$ is m -complete Σ_2^0 relative to \mathcal{O} .

1 Introduction

Our structures are all countable, with universe a subset of ω . The language of each structure is computable. When we measure the algorithmic complexity of a structure \mathcal{A} , we identify \mathcal{A} with its atomic diagram $D(\mathcal{A})$, and we identify the sentences of $D(\mathcal{A})$ with their Gödel numbers. Thus, \mathcal{A} is computable if $D(\mathcal{A})$ is a computable subset of ω . An *index* for a computable structure \mathcal{A} is a number e such that $\varphi_e = \chi_{D(\mathcal{A})}$.

The *Scott rank* of a countable structure \mathcal{A} , denoted by $SR(\mathcal{A})$, measures the model-theoretic complexity. We shall give a precise definition later. For a computable structure \mathcal{A} , $SR(\mathcal{A})$ is at most $\omega_1^{CK}+1$ (see Nadel [16]). The computable ordinals all have computable Scott rank. Moreover, there are computable ordinals of arbitrarily high computable Scott rank. Harrison [13] showed that there is a computable ordering of type $\omega_1^{CK}(1+\eta)$ (see also [18]). The Harrison ordering has Scott rank $\omega_1^{CK}+1$. There are some related computable structures, the *Harrison Boolean algebra* and the *Harrison Abelian p -groups*, which also have Scott rank $\omega_1^{CK}+1$ (see [9]).

Computable structures of Scott rank ω_1^{CK} took longer to find. Makkai [14] gave an example of an arithmetical structure of Scott rank ω_1^{CK} . In [11], it is shown that Makkai's example can be made computable. In [7], it is shown

*The authors are grateful for travel support from the National Foundation (under Grants #DMS 0075899 and #DMS 0554841) and the Charles L. Huisking Foundation. They are also grateful to the Russian Ministry of Education and Science for the grant which brought the second author to Notre Dame for the academic year 2006-2007.

that there is a computable tree of Scott rank ω_1^{CK} . Moreover, there is such a tree which shares with the Harrison ordering the feature that it is “strongly computably approximable”.

For a structure \mathcal{A} , the *index set*, denoted by $I(\mathcal{A})$, is the set of all indices for computable copies of \mathcal{A} . In [6], there are precise calculations of the complexity of the index sets for several kinds of structures, in particular, for all computable Abelian p -groups of length $< \omega^2$. For any computable structure \mathcal{A} , $I(\mathcal{A})$ is Σ_1^1 . If \mathcal{A} has computable Scott rank, then $I(\mathcal{A})$ is hyperarithmetical. It is an open problem whether the converse holds. There are a number of examples of computable structures \mathcal{A} such that the Scott rank is non-computable and $I(\mathcal{A})$ is m -complete Σ_1^1 . It is an open problem whether there is a computable structure \mathcal{A} of non-computable Scott rank such that $I(\mathcal{A})$ is *not* m -complete Σ_1^1 .

In the present paper, we study the index sets for *classes* of high rank structures. Our work is related to earlier work of W. White [20], [21], who calculated the complexity of the index sets for classes of computable structures having some properties of model-theoretic importance, such as homogeneity and saturation. We consider the following three classes.

1. $K_{\omega_1^{CK}} = \{\mathcal{A} : SR(\mathcal{A}) = \omega_1^{CK}\}$
2. $K_{\omega_1^{CK}+1} = \{\mathcal{A} : SR(\mathcal{A}) = \omega_1^{CK} + 1\}$
3. $K = K_{\omega_1^{CK}} \cup K_{\omega_1^{CK}+1}$

In Section 2, we will give some background on the complexity classes (Π_1^1 , etc.) and on Scott rank. In Section 3, we recall from [7] some special features of the tree T of Scott rank ω_1^{CK} together with a family of approximating trees $(T_a)_{a \in P}$ of computable Scott rank. In Section 4 we bound the complexity of the three index sets, showing that each has a definition of a certain form. We arrive at the fact that $I(K)$ is Σ_1^1 , $I(K_{\omega_1^{CK}})$ is Π_2^0 relative to \mathcal{O} , and $I(K_{\omega_1^{CK}+1})$ is Σ_2^0 relative to \mathcal{O} . In Section 5 we prove that our bounds are sharp—each index set is m -complete among sets having definitions of the same form.

2 Background

2.1 Π_1^1 sets

Kleene defined the *analytical hierarchy*, starting with computable relations on numbers and functions (from ω to ω) and closing under projection and complement. We need only the bottom part of this hierarchy. A relation $R(\bar{x}, f)$, with number variables \bar{x} and a function variable f , is *computable* if there is some e such that for all \bar{x} in ω and all $f \in \omega^\omega$,

$$\varphi_e^f(\bar{x}) \downarrow = \begin{cases} 1 & \text{if } R(\bar{x}, f) \\ 0 & \text{otherwise} \end{cases}$$

Definition 1. Let $S(\bar{x})$ be a relation

1. $S(\bar{x})$ is Π_1^1 if it can be expressed in the form $(\forall f) R(\bar{x}, f)$, where $R(\bar{x}, f)$ is computable,
2. $S(\bar{x})$ is Σ_1^1 if it can be expressed in the form $(\exists f) R(\bar{x}, f)$, where $R(\bar{x}, f)$ is computable,
3. $S(\bar{x})$ is Δ_1^1 if it is both Π_1^1 and Σ_1^1 .

If $S(\bar{x})$ is a k -place relation, we may consider the set S' of codes for k -tuples. It is clear that, S is Π_1^1 iff S' is Π_1^1 . The same is true for Σ_1^1 and Δ_1^1 relations. The next result gives familiar conditions equivalent to being Π_1^1 [17], [2]. We identify finite sequences with their codes.

Proposition 2.1 (Kleene). *The following are equivalent:*

1. S is Π_1^1 ,
2. there is a computable relation $R(n, u)$, on pairs of numbers, such that $n \in S$ iff $(\forall f) (\exists s) R(n, f \upharpoonright s)$,
3. there is a c.e. relation $R(n, u)$, on pairs of numbers, such that $n \in S$ iff $(\forall f) (\exists s) R(n, f \upharpoonright s)$,
4. there is a uniformly computable sequence of trees $(T_n)_{n \in \omega}$ such that $n \in S$ iff T_n has no path.

We would like to assign to each e a Π_1^1 set S for which e is an index. We consider W_e to be a set of pairs $\langle n, v \rangle$. Let R_e be the set of pairs (n, u) such that u is a finite sequence, and s is the length of u , then for all $t \leq s$ and $v = u \upharpoonright t$, $\langle n, v \rangle \notin W_{e,s}$. For each n , the set of u such that $(n, u) \in R_e$ is a computable tree $T_{e,n}$. We let e be an index for the Π_1^1 set S consisting of those n such that $T_{e,n}$ has no path. Note that from e , we can compute an index for the sequence of trees $(T_{e,n})_{n \in \omega}$.

Definition 2 (Universal Π_1^1 enumeration relation). *Let $E_{\Pi_1^1}(e, n)$ be the set of pairs (e, n) such that n is in the Π_1^1 set with index e .*

Fact. The relation $E_{\Pi_1^1}$ is Π_1^1 .

2.2 Computable infinitary formulas

Roughly speaking, the computable infinitary formulas are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are c.e. While these formulas are infinitely long, they still seem comprehensible. The formulas are all in “normal form”, with the negations brought inside. They are classified as computable Σ_α , or computable Π_α , for various computable ordinals α . Taken all together, the computable infinitary formulas are essentially the same as the formulas in the least admissible fragment of $L_{\omega_1\omega}$. What makes the computable infinitary formulas useful is the classification.

Definition 3.

1. $\varphi(\bar{x})$ is computable Σ_0 and Π_0 if it is finitary quantifier-free,
2. for $\alpha > 0$
 - (a) $\varphi(\bar{x})$ is computable Σ_α if it is a c.e. disjunction of formulas of the form $(\exists \bar{u}) \psi(\bar{x}, \bar{u})$, where ψ is computable Π_β for some $\beta < \alpha$,
 - (b) $\varphi(\bar{x})$ is computable Π_α if it is a c.e. conjunction of formulas of the form $(\forall \bar{u}) \psi(\bar{x}, \bar{u})$, where ψ is computable Σ_β for some $\beta < \alpha$.

We write $neg(\varphi)$ for the effectively determined formula in dual form that is logically equivalent to the negation of φ . (If φ is computable Σ_α , then $neg(\varphi)$ is computable Π_α , and vice versa.) For more information on computable infinitary formulas, see [2]. The computable infinitary formulas match up with the hyperarithmetical hierarchy. Computable infinitary formulas have indices which depend on ordinal notation.

The hyperarithmetical sets also have indices. Given a notation a for α , there is a naturally associated complete Δ_α^0 oracle, which we call Δ_a^0 . If $\alpha \geq \omega$, we choose $H(a)$, while if α is a finite $n \geq 1$, we choose $\emptyset^{(n)}$ —this is $H(b)$, where b is the notation for $n - 1$. A set is Σ_α^0 , or Π_α^0 , or Δ_α^0 , if it is c.e., co-c.e., or computable, relative to Δ_a^0 , where a is a notation for α . By a result of Spector, this is independent of the notation a .

Proposition 2.2. *If φ is computable Σ_α , or computable Π_α , then satisfaction of φ in computable structures is Σ_α^0 , or Π_α^0 .*

This proposition is true with a great deal of uniformity, over the formula, the structure, even the ordinal, or notation. Given an index for a computable infinitary formula, we see that the formula is computable Σ_α or computable Π_α , and we are given a notation a for α . Given an index for a computable structure, we can pass effectively to an index for the Σ_α^0 , or Π_α^0 relation defined by the formula.

There is a version of Compactness for computable infinitary formulas. For a proof of the version stated below, and various corollaries, see [2].

Theorem 2.3 (Barwise-Kreisel Compactness). *Let Γ be a Π_1^1 set of computable infinitary sentences. If every Δ_1^1 set $\Gamma' \subseteq \Gamma$ has a model, then Γ has a model.*

We can use Barwise-Kreisel Compactness to produce computable structures.

Corollary 2.4. *Let Γ be a Π_1^1 set of computable infinitary sentences. If every Δ_1^1 subset has a computable model, then the whole set has a computable model.*

Any computable (or hyperarithmetical) structure has some saturation properties. The next two results are familiar consequences of Barwise-Kreisel Compactness.

Corollary 2.5 (Ressayre). *Let \mathcal{A} be a computable structure and let \bar{a} be a tuple in \mathcal{A} . Let $\Gamma(\bar{a}, x)$ be a Π_1^1 set of formulas. If every Δ_1^1 subset is satisfied in \mathcal{A} , then the whole set is satisfied in \mathcal{A} .*

Corollary 2.6. *Suppose \mathcal{A} and \mathcal{B} are computable structures satisfying the same computable infinitary sentences. Then $\mathcal{A} \cong \mathcal{B}$.*

2.3 Tree rank and approximations for Π_1^1 sets

We consider trees which are isomorphic to subtrees of $\omega^{<\omega}$. We define *tree rank* of nodes x in a tree T .

Definition 4 (Tree rank).

1. $tr(x) = 0$ if x has no successor.
2. $tr(x) = \alpha$ if all successors x' of x have ordinal tree rank, and α is least ordinal greater than all of these ordinals.
3. $tr(x) = \infty$ if the definition above does not assign ordinal tree rank to x .

We let $tr(T)$ be the tree rank of the top node \emptyset .

Fact: Suppose T is a computable tree. If T has a path, then $tr(T) = \infty$, and otherwise $tr(T)$ is a computable ordinal.

We can write natural computable infinitary sentences describing the tree rank. There is a computable Π_2 sentence saying that the tree rank is at least ω , and a computable $\Pi_{2\alpha}$ sentence saying that the tree rank is at least $\omega\alpha$. Moreover, given a notation for α , we can effectively find such a sentence.

For a Π_1^1 set S , we have a natural family of approximations S_α obtained in the following way. Let $(T_n)_{n \in \omega}$ be the uniformly computable sequence of trees associated with our index e for S (as in the paragraph after Proposition 2.1), such that $n \in S$ iff T_n has no path. Given α , let $\hat{\alpha}$ be greatest such that $2\hat{\alpha} \leq \alpha$. Given a notation a for α , we can effectively find the notation \hat{a} for $\hat{\alpha}$ such that $\hat{a} \leq_{\mathcal{O}} a$. We let $n \in S_\alpha$ iff $tr(T_n) \not\leq \omega\hat{\alpha}$. The condition $tr(T_n) \not\leq \omega\hat{\alpha}$ is Σ_α^0 , uniformly in n and the notation for α . By this, we mean that if Δ_a^0 is the complete Δ_α^0 oracle associated with the notation a for α , then we can apply a uniform procedure to enumerate those n such that $(T_n) \not\leq \omega\hat{\alpha}$.

2.4 Scott rank

Scott rank comes from the Scott Isomorphism Theorem. Scott [19] proved that for any (countable) structure \mathcal{A} (for a countable language L), there is an $L_{\omega_1\omega}$ sentence σ whose countable models are just the isomorphic copies of \mathcal{A} . In the proof, Scott associated a countable ordinal to each tuple in the structure \mathcal{A} , and to the structure itself. We begin with a family of equivalence relations on the tuples in the structure \mathcal{A} .

Definition 5 (Equivalence relations \equiv^α).

1. $\bar{a} \equiv^0 \bar{b}$ if \bar{a} and \bar{b} satisfy the same atomic formulas,
2. for $\alpha > 0$, $\bar{a} \equiv^\alpha \bar{b}$ if for all $\beta < \alpha$, for all \bar{c} , there exists \bar{d} , and for all \bar{d} , there exists \bar{c} , such that $\bar{a}, \bar{c} \equiv^\beta \bar{b}, \bar{d}$.

Our equivalence relations differ slightly from Scott's—we extend by tuples \bar{c} and \bar{d} , where he extended by single elements c and d .

Definition 6 (Scott rank).

1. For each tuple \bar{a} in the structure \mathcal{A} , the Scott rank of \bar{a} is the least ordinal β such that for all \bar{b} , $\bar{a} \equiv^\beta \bar{b}$ implies $(\mathcal{A}, \bar{a}) \cong (\mathcal{A}, \bar{b})$.
2. The Scott rank of the structure \mathcal{A} , denoted by $SR(\mathcal{A})$, is the least ordinal α greater than the Scott ranks of all tuples in \mathcal{A} .

Our definition agrees with Scott's, and with a different-looking definition used by J. Millar and Sacks [15], on which computable structures have computable rank, which have rank ω_1^{CK} , and which have rank $\omega_1^{CK} + 1$. The result below was stated explicitly in [11], but well-known before. For a proof, see [7] pp. 283-284.

Theorem 2.7 (Folklore).

1. $SR(\mathcal{A})$ is computable if there is a computable ordinal α such that the orbits of all tuples are defined by computable Π_α formulas,
2. $SR(\mathcal{A}) = \omega_1^{CK}$ if the orbits of all tuples are defined by computable infinitary formulas, but there is no computable ordinal α , as in 1,
3. $SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is some tuple whose orbit is not defined by any computable infinitary formula.

The following condition comes from another definition of rank, due to Barwise [3]. The *Barwise rank*, $BR(\mathcal{A})$ is the least ordinal α such that for all tuples \bar{a} and \bar{b} in \mathcal{A} , if $\bar{a} \equiv^\alpha \bar{b}$, then $\bar{a} \equiv^{\alpha+1} \bar{b}$.

Note: For a given tuple \bar{a} in \mathcal{A} , the fact that for all \bar{b} , $\bar{a} \equiv^\alpha \bar{b}$ implies $\bar{a} \equiv^{\alpha+1} \bar{b}$, does not imply that \bar{a} has Scott rank $\leq \alpha$.

The following is well-known, and not difficult to prove.

Proposition 2.8 (Folklore). *Suppose \mathcal{A} is a computable structure.*

1. $BR(\mathcal{A}) < \omega_1^{CK}$ iff $SR(\mathcal{A}) < \omega_1^{CK}$.
2. $BR(\mathcal{A}) = \omega_1^{CK}$ iff $SR(\mathcal{A}) = \omega_1^{CK}$ or $\omega_1^{CK} + 1$.

2.5 Strong computable approximability

Definition 7. Let \mathcal{A} be a computable structure of non-computable Scott rank. We say that \mathcal{A} is strongly computably approximable if for any Σ_1^1 set S , there is a uniformly computable sequence $(C_n)_{n \in \omega}$ such that for $n \in S$, $C_n \cong \mathcal{A}$ and for $n \notin S$, C_n has computable Scott rank.

The following is well known (see [8]).

Proposition 2.9. *The Harrison ordering is strongly computably approximable, where the approximating structures are well orderings.*

Sketch of proof. Let S be a Σ_1^1 set. Let $(T_n)_{n \in \omega}$ be a uniformly computable sequence of trees such that $n \in S$ iff T_n has a path. Let T^K be a computable tree with a path but with no hyperarithmetical path. (We could use Barwise Compactness to produce such a tree, although Kleene did it by other means.) Given an index for T_n , we pass effectively to an index for a tree T_n^K such that a path through T_n^K represents a path through T_n and a path through T^K . Each element of T_n^K is a sequence $\langle n_0, m_0 \rangle, \langle n_1, m_1 \rangle, \dots, \langle n_r, m_r \rangle$, where $\sigma = n_0, n_1, \dots, n_r \in T_n$, and $\tau = m_0, m_1, \dots, m_r \in T^K$. From an index for T_n^K , we pass effectively to an index for a tree T'_n , which has infinitely many nodes at level 1, with a copy of T_n^K below each. Let L_n be the Kleene-Brouwer ordering of T'_n . If $n \in S$, then L_n is a Harrison ordering, and if $n \notin S$, it is a well ordering. □

In [7], it is shown that there is a computable tree of Scott rank ω_1^{CK} which is strongly computably approximable. We say more about the tree and its approximations in the next section. In [4] and [5], rank preserving computable embeddings are used to show that there are strongly computably approximable structures of Scott ranks ω_1^{CK} and $\omega_1^{CK} + 1$ in the following classes: trees, graphs, fields of characteristic 0, linear orderings, Boolean algebras.

3 A special family of trees

In [7], there is a construction of a computable tree T of Scott rank ω_1^{CK} , together with a uniformly computable family of approximating trees $(T_a)_{a \in P}$. The set P is the maximal well ordered initial segment in a Harrison ordering. Moreover, the Harrison ordering has the feature that we can effectively recognize successor and limit elements, and we can effectively pass from a successor element a to its predecessor, and from a limit element a to an index for a strictly increasing sequence with limit a . There is a partial computable function f that maps P onto a path through \mathcal{O} , such that if $a \in P$, then $f(a)$ is a notation for the order type of $\text{pred}(a)$. We identify P with the path, and for $a \in P$, we write $|a|$ for the ordinal for which a is a notation; i.e., the order type of $\text{pred}(a)$.

The trees T and $(T_a)_{a \in P}$ have the following three properties, which will be important later.

Property I. Given $a \in P$, we can effectively find a computable infinitary Scott sentence for T_a .

Property II. For each $a \in P$, the $\Sigma_{|a|}$ sentences true in T_a are true in T .

Property III. Given $a < b$ in P , and an index for a $\Pi_{|a|}^0$ set S , we can find an index for a uniformly computable sequence $(\mathcal{C}_n)_{n \in \omega}$ such that

$$\mathcal{C}_n \cong \begin{cases} T_a & \text{if } n \in S \\ T_b & \text{otherwise} \end{cases}$$

We offer partial explanations for these three properties. For more details, the reader should see [7].

Definition 8 (rank homogeneity). *A tree T is rank homogeneous if it satisfies the following conditions:*

1. *for any node x at level n (of tree rank an ordinal or ∞), for any $\beta < tr(x)$ such that there is a node of tree rank β at level $n+1$, x has infinitely many successors of tree rank β ,*
2. *if $tr(x) = \infty$, then x has infinitely many successors of tree rank ∞ .*

If T is a rank homogeneous tree, then the isomorphism type of T is determined by the set of tree ranks that occur at the various levels. If T is a computable tree, then either $tr(T) = \infty$ or else $tr(T)$ is a computable ordinal. We can describe the Scott ranks of computable rank homogeneous trees.

Proposition 3.1. *Suppose T is a computable rank homogeneous tree.*

1. *$SR(T)$ is computable if there is a computable bound on the ordinal tree ranks that occur,*
2. *$SR(T)$ is ω_1^{CK} if for each level, there is a computable bound on the ordinal tree ranks that occur, but there is no computable bound over-all,*
3. *$SR(T)$ is $\omega_1^{CK} + 1$ if there is some level with no computable bound on the ordinal tree ranks.*

The tree T_a is rank homogeneous, and with tree rank at most $\omega(|a| + 1)$. This explains how we can pass effectively from a to a computable infinitary Scott sentence for T_a . Given a , we can effectively find T_a . Using $\Delta_{2|a|+2}^0$, we can determine which tree ranks occur at each level. Then we can effectively write a Scott sentence saying that the tree is rank homogeneous, with a certain set of tree ranks at each level. There are familiar tricks for turning $\Sigma_{2|a|+2}^0$ disjunctions, or conjunctions, into c.e. disjunctions, or conjunctions. Thus, we pass effectively from a to a computable infinitary Scott sentence, We have explained Property I.

We define binary relations \preceq_β (or \preceq_b , where $|b| = \beta$) such that

$$(a, \bar{x}) \preceq_\beta (a', \bar{x}')$$

iff the subtree of T_a generated by \bar{x} is isomorphic to the subtree of $T_{a'}$ generated by \bar{x}' and for corresponding elements z and z' of these finite subtrees, either the tree ranks match or else both are $\geq \omega \cdot \beta$. The trees T_a are constructed so that for all n , for all $a \in P$, the tree ranks $< \omega \cdot |a|$ that occur at level n in T_a and T are the same. This implies that for all $b \geq a$, $(a, \emptyset) \preceq_{|a|} (b, \emptyset)$ and $(b, \emptyset) \preceq_{|a|} (a, \emptyset)$. The relations \preceq_β have the “back-and-forth property”. This is defined below.

Definition 9 (Back-and-forth family of relations). *Let $\mathcal{A}_1, \mathcal{A}_2$ be structures. We consider pairs (i, \bar{a}) , where \bar{a} is a tuple in \mathcal{A}_i . Let $(\preceq_\beta)_{\beta < \alpha}$ be a family of relations on these pairs. The family has the back-and-forth property if the following conditions hold.*

1. *If $(i, \bar{a}) \leq_0 (j, \bar{b})$, then the quantifier-free formulas true of \bar{a} in \mathcal{A}_i are true of \bar{b} in \mathcal{A}_j ,*
2. *For $\beta > 0$, if $(i, \bar{a}) \leq_\beta (j, \bar{b})$, then for all $\gamma < \beta$ and all \bar{d} , there exists \bar{c} such that $(j, \bar{b}, \bar{d}) \leq_\gamma (i, \bar{a}, \bar{c})$.*

The following is easy to prove, and well-known. It is essentially due to Carol Karp.

Proposition 3.2. *Let $\mathcal{A}_1, \mathcal{A}_2$ be structures, and let $(\preceq_\beta)_{\beta < \alpha}$ be a family of relations with the back-and-forth property, as above.*

1. *If $(i, \bar{x}) \leq_\beta (j, \bar{y})$, then all Π_β formulas true of \bar{x} in \mathcal{A}_i are true of \bar{y} in \mathcal{A}_j .*
2. *If for each $\beta < \alpha$ and each \bar{x} in \mathcal{A}_1 , there exists \bar{y} in \mathcal{A}_2 such that $(i, \bar{x}) \leq_\beta (j, \bar{y})$, then all Σ_α sentences true in \mathcal{A}_1 are true in \mathcal{A}_2 .*

We can apply Proposition 3.2 to the special trees $(T_a)_{a \in P}$, and the back-and-forth relations $(\preceq_{|b|})_{b \in P}$. Suppose that $|a| = \alpha$. Since $(a, \emptyset) \preceq_\alpha (b, \emptyset)$ and $(b, \emptyset) \preceq_\alpha (a, \emptyset)$, T_a and T_b satisfy the same Σ_α sentences. The tree T also satisfies these sentences. We have explained Property II.

For Property III, we need the following definition. Here we identify computable ordinals with their notations in P .

Definition 10 (α -friendly pair). *A pair of structures $\mathcal{A}_1, \mathcal{A}_2$ is α -friendly under the back-and-forth family $(\preceq_\beta)_{\beta < \alpha}$ provided that the structures are computable, and the relations \preceq_β are uniformly c.e.*

The trees T_a are equipped with functions $f_a : T_a \rightarrow P \times \omega$, computable uniformly in a , indicating the tree rank. For $x \in T_a$, $f_a(x) = (b, n)$ indicates that $tr(x) = \omega|b| + n$. This implies that for any a, b, c in P , the pair T_a, T_b is $|c|$ -friendly. This puts us in a position to use the result below (see [1], [2]). Again we identify computable ordinals with their notations in P .

Theorem 3.3 (Pairs of structures). *Suppose that \mathcal{A}_1 and \mathcal{A}_2 form an α -friendly pair, under the relations $(\leq_\beta)_{\beta < \alpha}$. Suppose further that for any $\beta < \alpha$ and any $\bar{x} \in \mathcal{A}_1$, there exists $\bar{y} \in \mathcal{A}_2$ such that $(\mathcal{A}_1, \bar{x}) \leq_\beta (\mathcal{A}_2, \bar{y})$. Then for any Π_α^0 set S , there is a uniformly computable sequence $(\mathcal{C}_n)_{n \in \omega}$ such that*

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}_1 & \text{if } n \in S \\ \mathcal{A}_2 & \text{otherwise} \end{cases}$$

Moreover, we can compute an index for the uniformly computable sequence $(\mathcal{C}_n)_{n \in \omega}$ effectively from the notation a for α , computable indices for $\mathcal{A}_1, \mathcal{A}_2$, an index for the uniformly c.e. sequence of relations $(\leq_\beta)_{\beta < \alpha}$, and an index for a Π_α^0 set S .

This explains Property III.

4 Bounds on $I(K)$, $I(K_{\omega_1^{CK}})$, and $I(K_{\omega_1^{CK+1}})$

In this section, we show that $I(K)$ is Σ_1^1 , that $I(K_{\omega_1^{CK}})$ has a definition of the form $(\forall)(\exists)(\Pi_1^1 - \Pi_1^1)$, and that $I(K_{\omega_1^{CK+1}})$ has a definition of the form $(\exists)(\forall)(\Pi_1^1 \rightarrow \Pi_1^1)$. We describe the classes defined in this way, and we indicate how the last two forms match more familiar classes arithmetical over \mathcal{O} . We are grateful to Carl Jockusch for pointing out this connection.

4.1 The classes D_n and I_n

Let D_n be the family of sets defined by a sequence of n alternating number quantifiers, ending in (\exists) , followed by a relation of the form $\Pi_1^1 - \Pi_1^1$. Let I_n be the family of sets defined by a sequence of n alternating number quantifiers, ending in (\forall) , followed by a relation of the form $\Pi_1^1 \rightarrow \Pi_1^1$. The sets and relations in I_n are the complements of those in D_n . It is easy to see that if Q is in D_n , or I_n , and $R \leq_m Q$, then R is in D_n , or I_n . When we calculate the complexity of $I(K_{\omega_1^{CK}})$ and $I(K_{\omega_1^{CK+1}})$, it will be convenient for us to think of the classes D_2 and I_2 . We show that $I(K_{\omega_1^{CK}})$ is m -complete among sets in D_2 and $I(K_{\omega_1^{CK+1}})$ is m -complete among sets in I_2 .

Proposition 4.1 (Jockusch). *For odd $n \geq 1$, the sets in D_n are exactly those $\Sigma_n^0(\mathcal{O})$, and the sets in I_n are exactly those $\Pi_n^0(\mathcal{O})$. For even $n \geq 1$, the sets in D_n are exactly those $\Pi_n^0(\mathcal{O})$, and those in I_n are exactly those $\Sigma_n^0(\mathcal{O})$.*

Proof. We give the proof for $n = 1$. It is enough to show that $S \in D_1$ iff S is c.e. relative to \mathcal{O} , since this implies $\bar{S} \in I_1$ iff \bar{S} is co-c.e. relative to \mathcal{O} . If $S \in D_1$, then it is clear that S is c.e. relative to \mathcal{O} . Suppose S is c.e. relative to \mathcal{O} . Then $S \leq_m \mathcal{O}'$. To show that $S \in D_1$, it is enough to show that \mathcal{O}' is in D_1 . Now, $e \in \mathcal{O}'$ iff there exists σ such that $\varphi_e^\sigma(e) \downarrow$, where σ is a finite initial segment of the characteristic function of \mathcal{O} . This has the form $(\exists)(\Pi_1^1 - \Pi_1^1)$. \square

From Proposition 4.1, we get non-collapse. For $n \geq 1$, any set in D_n or I_n is in both D_{n+1} and I_{n+1} . There is a set in $D_n - I_n$, so the complement is in $I_n - D_n$. We add two further simple facts to complete the picture.

Proposition 4.2. $D_0 \subseteq I_1$ and $I_0 \subseteq D_1$.

Proof. It is enough to show that $D_0 \subseteq I_1$. (We get the fact that $I_0 \subseteq D_1$ taking complements.) Let T^P be a computable tree with a path, and let T^N be a computable tree with no path. Let $S = S^1 - S^2$, where the S^i are Π_1^1 sets. Let $(T_n^i)_{n \in \omega}$ be uniformly computable families of trees such that $n \in S^i$ iff T_n^i has no path. We define uniformly computable families of trees $T'_{n,x}$ and $T''_{n,x}$ such that

$$\begin{cases} T'_{n,x} = T^N \ \& \ T''_{n,x} = T_n^1 & \text{for even } x \\ T'_{n,x} = T_n^2 \ \& \ T''_{n,x} = T^P & \text{for odd } x \end{cases}$$

We have $n \in S^1$ iff T_n^1 has no path iff for all even x such that $T'_{n,x}$ has no path, $T''_{n,x}$ also has no path. Similarly, $n \notin S^2$ iff T_n^2 has a path iff for all odd x such that $T'_{n,x}$ has a path, $T''_{n,x}$ also has a path; equivalently, for all odd x such that $T'_{n,x}$ has no path, $T''_{n,x}$ also has no path. Therefore, $n \in S^1 - S^2$ iff for all x such that $T'_{n,x}$ has no path, $T''_{n,x}$ also has no path. Therefore, S is in I_1 . \square

Proposition 4.3. *There is a set S in $D_0 - I_0$. Then the complement \bar{S} is in $I_0 - D_0$.*

Proof. We note that from the Π_1^1 uniform enumeration relation $E_{\Pi_1^1}$ for Π_1^1 sets, we get a D_0 enumeration relation E_0 for D_0 sets. We let $E_{D_0}(\langle e_1, e_2 \rangle, n)$ iff $E_{\Pi_1^1}(e_1, n) \ \& \ \neg E_{\Pi_1^1}(e_2, n)$. This clearly has the right form. Let $C(e, n)$ be the complementary relation, so $C(e, n)$ iff $\neg E_{D_0}(e, n)$. This is in I_0 . We show that it is not in D_0 . If the relation C (on two variables) is in D_0 , then the set of n such that $C(n, n)$ holds is also in D_0 . Say e is a D_0 index for this set. Then for all n , $E_{D_0}(e, n)$ iff $C(n, n)$. In particular, $C(e, e)$ iff $E_{D_0}(e, e)$. However, from the definition of $C(e, n)$ as the complement of the enumeration relation, we have $C(e, e)$ iff $\neg E_{D_0}(e, e)$. This is a contradiction. \square

4.2 Definitions for the index sets

Our first goal is to show that $I(K)$ is Σ_1^1 . Let \mathcal{A}_e be the computable structure, if any, with computable index e . Let I be the set of indices for computable structures. Clearly, I is arithmetical (Π_2^0).

Lemma 4.4. *There is a Π_1^1 set Γ of computable infinitary sentences such that the computable models of Γ are exactly those of non-computable Scott rank.*

Proof. First, for each β and n , we can find a computable infinitary formula saying of n -tuples \bar{x} and \bar{y} (in an arbitrary structure for our language) that $\bar{x} \equiv^\beta \bar{y}$. For each computable ordinal α , we can find a computable infinitary sentence saying that there are tuples \bar{a} and \bar{b} such that $\bar{a} \equiv^\alpha \bar{b}$ but not $\bar{a} \equiv^{\alpha+1} \bar{b}$. These sentences say that the Barwise rank is not computable. By Proposition 2.8, they also guarantee non-computable Scott rank. \square

Satisfaction of computable Σ_α (or computable Π_α) formulas in computable structures is Σ_α^0 (or Π_α^0), with enough uniformity that (for a fixed computable language L), the relation “ φ is a computable infinitary sentence true in the computable structure \mathcal{A}_e ” is Π_1^1 . We can pass effectively from φ to $neg(\varphi)$, so the relation “ $neg(\varphi)$ is true in \mathcal{A}_e ” is also Π_1^1 .

Lemma 4.5. *Let Γ be a Π_1^1 set of computable infinitary sentences, and let K^Γ be the class of all computable models of Γ . Then $I(K^\Gamma)$ is Σ_1^1 .*

Proof. We consider the complement of $I(K^\Gamma)$. First, let I be the set of all indices for computable L -structures. This is Π_2^0 . We have $e \notin I(K^\Gamma)$ iff either $e \notin I$ or else $e \in I$ and $(\exists \varphi)(\varphi \in \Gamma \ \& \ \mathcal{A}_e \models neg(\varphi))$. The second disjunct has the form $(\exists)(\Pi_1^1 \ \& \ \Pi_1^1)$. The intersection of Π_1^1 relations is Π_1^1 , so we have the form $(\exists) \Pi_1^1$. The projection of a Π_1^1 relation on a number variable is Π_1^1 . The Π_2^0 disjunct can be absorbed, so the complement of $I(K^\Gamma)$ is Π_1^1 . Therefore, $I(K^\Gamma)$ itself is Σ_1^1 . \square

Lemmas 4.4 and 4.5 yield the following.

Proposition 4.6. *$I(K)$ is Σ_1^1 .*

Next, we give upper bounds for the complexity of $I(K_{\omega_1^{CK}})$ and $I(K_{\omega_1^{CK+1}})$.

Proposition 4.7.

1. $I(K_{\omega_1^{CK}})$ is in D_2 , so it is $\Pi_2^0(\mathcal{O})$.
2. $I(K_{\omega_1^{CK+1}})$ is in I_2 , so it is $\Sigma_2^0(\mathcal{O})$.

Proof. We say $e \in I(K_{\omega_1^{CK}})$ in the following way. First, we say that it is in $I(K)$. This is Σ_1^1 . Next, we say that for each n and each n -tuple \bar{x} , there is an ordinal α such that for all n -tuples \bar{y} , if $\bar{x} \equiv^\alpha \bar{y}$ in \mathcal{A}_e , then there is an automorphism of \mathcal{A}_e taking \bar{x} to \bar{y} . Given e , n , and an ordinal α , or a notation a for α , we can find a Π_1^1 index $g(e, a, n)$ for the relation “ $\bar{x} \equiv^\alpha \bar{y}$ in \mathcal{A}_e ”. We can also find a Σ_1^1 index $i(e, n)$ for the relation “there is an automorphism of \mathcal{A}_e taking \bar{x} to \bar{y} ”. Let $E_{\Sigma_1^1}$ be a Σ_1^1 enumeration relation for Σ_1^1 sets, and let $E_{\Pi_1^1}$ be a Π_1^1 enumeration relation for Π_1^1 sets.

To say that $e \in I(K_{\omega_1^{CK}})$, we write

$$e \in I(K) \ \& \ \bigwedge_n (\forall \bar{x}) (\exists a) [a \in \mathcal{O} \ \& \ (\forall \bar{y}) (E_{\Pi_1^1}(g(e, a, n), \bar{x}, \bar{y}) \rightarrow E_{\Sigma_1^1}(i(e, n), \bar{x}, \bar{y}))].$$

This has the form $\Sigma_1^1 \ \& \ (\forall) (\exists) [\Pi_1^1 \ \& \ (\forall) (\Pi_1^1 \rightarrow \Sigma_1^1)]$. Simplifying, we get the form $\Sigma_1^1 \ \& \ (\forall) (\exists) (\Pi_1^1 \ \& \ \Sigma_1^1)$, and then $(\forall) (\exists) (\Pi_1^1 \ \& \ \Sigma_1^1)$, or $(\forall) (\exists) (\Pi_1^1 - \Pi_1^1)$. Therefore, $I(K_{\omega_1^{cK}}) \in D_2$. To say that $e \in I(K_{\omega_1^{cK+1}})$, we say that

$$e \in K \ \& \ e \notin I(K_{\omega_1^{cK}}) .$$

We have seen that $I(K)$ is Σ_1^1 , and $I(K_{\omega_1^{cK}}) \in D_2$ so the complement is in I_2 . From this, it follows that $I(K_{\omega_1^{cK+1}}) \in I_2$. □

5 Sharpness of the Bounds

We have shown that $I(K)$ is Σ_1^1 , $I(K_{\omega_1^{cK}})$ is $\Pi_2^0(\mathcal{O})$, and $I(K_{\omega_1^{cK+1}})$ is $\Sigma_2^0(\mathcal{O})$. In this section, we show that each of these index sets is m -complete among sets in its definability class. This establishes that our definitions are best possible.

Theorem 5.1. $I(K)$ is m -complete Σ_1^1 .

Proof. By Proposition 4.6, $I(K)$ is Σ_1^1 . For completeness, take the Harrison ordering \mathcal{H} , or some other structure \mathcal{A} which is strongly computably approximable. For any Σ_1^1 set S , there is a uniformly computable sequence $(\mathcal{C}_n)_{n \in \omega}$ such that if $n \in S$, then $\mathcal{C}_n \cong \mathcal{A}$, so $\mathcal{C}_n \in K$, and if $n \notin S$, then \mathcal{C}_n has computable Scott rank, so $\mathcal{C}_n \notin K$. □

We turn to the sets $I(K_{\omega_1^{cK}})$ and $I(K_{\omega_1^{cK+1}})$. We use the following definition.

Definition 11. Let $A \subseteq B$ and let S be another set. We say $S \leq_m A$ within B if there is a computable function $f : \omega \rightarrow B$ such that $n \in S$ iff $f(n) \in A$.

Before proving that $I(K_{\omega_1^{cK}})$ is m -complete $\Pi_2^0(\mathcal{O})$ and $I(K_{\omega_1^{cK+1}})$ is m -complete $\Sigma_2^0(\mathcal{O})$, we give the following weaker result.

Theorem 5.2.

1. For all $S \in D_1$, $S \leq_m I(K_{\omega_1^{cK}})$ within $I(K)$.
2. For all $S \in I_1$, $S \leq_m I(K_{\omega_1^{cK+1}})$ within $I(K)$.

Proof. Let S be a set in D_1 . Say S^1 and S^2 are Π_1^1 sets such that $n \in S$ iff $(\exists x) (n, x) \in S^1 - S^2$. We shall produce a uniformly computable sequence $(\mathcal{A}_n)_{n \in \omega}$ such that if $n \in S$, then $\mathcal{A}_n \in K_{\omega_1^{cK}}$, and if $n \notin S$, then $\mathcal{A}_n \in K_{\omega_1^{cK+1}}$. We want approximations for S , so that if $n \in S$ then for all sufficiently large α , $n \in S_\alpha$, and if $n \notin S$, then for arbitrarily large α , $n \notin S_\alpha$.

If we had $n \in S$ iff $(\exists x) (n, x) \in S^1 - S^2$, where S^1 and S^2 are c.e., then we would approximate S as follows. At each stage s such that $(\exists x) (n, x) \in S_s^1 - S_s^2$, we designate the first as $x(n, s)$. We let $n \in S_s$ iff there is a currently designated

$x = x(n, s)$ and the previous one has not just entered S^2 ; i.e., we do not have $x' = x(n, s - 1)$ and $x' \in S_s^2$. We do the same here, thinking of ordinals as stages. At each stage α such that $(\exists x)(n, x) \in S_\alpha^1 - S_\alpha^2$, we designate the first as $x(n, \alpha)$. We let $n \in S_\alpha$ iff there is a current designated $x(n, \alpha)$ and it is not the case that for some $\beta < \alpha$ and x' , for all $\beta \leq \gamma < \alpha$, $x(n, \gamma) = x'$ and $x' \in S_\alpha^2$.

Lemma 5.3.

1. If $n \in S$, then $(\exists \alpha)(\forall \beta > \alpha) n \in S_\beta$
2. If $n \notin S$, then $(\forall \alpha)(\exists \beta > \alpha) n \in S_\beta$.

Proof. We identify the ordinals with their notations. First, suppose $n \in S$, so $(\exists x)(n, x) \in S^1 - S^2$. Let x be the first such that $(n, x) \in S^1 - S^2$. Take a stage α by which (n, x) has entered S^1 and for all $x' < x$ that are ever designated, (n, x') has entered S^2 . For all $\beta \geq \alpha$, $x(n, \beta) = x$. Then for all $\beta \geq \alpha$, $(n, x) \in S_\beta^1 - S_\beta^2$. Then for all $\beta > \alpha$, $n \in S_\beta$.

Next, suppose $n \notin S$, so $(\forall x)(n, x) \notin S^1 - S^2$. Let α be given. It may be that for some $\beta > \alpha$, there is no designated $x(n, \beta)$, and then $n \notin S_\beta$. Suppose there is always a designated x . Let x be the first designated at some stage $\beta \geq \alpha$. Then $(n, x) \in S_\beta^1 - S_\beta^2$. Since $(n, x) \notin S^1 - S^2$, there must be some stage $\beta' > \beta$ at which (n, x) enters S^2 . For $\beta \leq \gamma < \beta'$, $x(n, \gamma) = x$. Then $n \notin S_{\beta'}$. \square

We consider the oracle needed to determine whether $n \in S_\beta$.

Lemma 5.4. *Suppose $|a| = |b| + 2$. Then using Δ_a^0 , we can decide whether $n \in S_\beta$. Moreover, the procedure is uniform.*

Proof. First, suppose β is a successor, say $\beta = \gamma + 1$, then $n \in S_\beta$ iff there exists x such that $(n, x) \in S_\beta^1 - S_\beta^2$ and if there is a first x' such that $(n, x') \in S_\gamma^1 - S_\gamma^2$, then $x' \notin S_\beta^2$. Using $\Delta_{\beta+1}^0$, we can decide whether a given (n, x) is in $S_\beta^1 - S_\beta^2$. Using $\Delta_{\beta+2}^0$, we can decide whether there is any such (n, x) . Using $\Delta_{\beta+1}^0$, we can decide whether there is some x' such that $(n, x') \in S_\gamma^1 - S_\gamma^2$, and if so, we can find the first such x' . Using $\Delta_{\beta+1}^0$, we can decide whether $(n, x') \in S_\beta^2$. Thus, using Δ_α^0 , we can decide whether $n \in S_\beta$.

Now suppose β is a limit. Then $n \in S_\beta$ iff there exists x such that $(n, x) \in S_\beta^1 - S_\beta^2$ and there do not exist x' and $\beta' < \beta$ such that for all $\beta' \leq \gamma < \beta$, x' is first with $(n, x') \in S_\gamma^1 - S_\gamma^2$. For a given $\gamma < \beta$, we can determine using $\Delta_{\gamma+1}^0$ whether $x' = x(n, \gamma)$. Using $\Delta_{\beta+1}^0$, we can determine, for a given x' and $\beta' < \beta$, whether $x(n, \gamma) = x'$ for all $\beta' \leq \gamma < \beta$. Using $\Delta_{\beta+2}^0$, we can determine whether there exist x' and $\beta' < \beta$ with this feature. Then, as above, we see that using Δ_α^0 , we can decide whether $n \in S_\beta$. \square

We describe the desired sequence of structures $(\mathcal{A}_n)_{n \in \omega}$. For each n , \mathcal{A}_n will have an equivalence relation \sim partitioning the universe into infinitely many

infinite classes. In addition, \mathcal{A}_n will have a binary relation which is the union of relations putting a tree structure on each \sim -equivalence class. We use the special tree T of Scott rank ω_1^{CK} and approximations $(T_a)_{a \in P}$, of computable Scott rank, constructed in [7], with properties described in Section 3.

Property I: Given $a \in P$, we can effectively find a computable infinitary Scott sentence for T_a .

Property II: For each $a \in P$, the computable $\Sigma_{|a|}$ sentences true in T_a are true in T .

Property III: Given $a < b$ in P , and an index for a $\Pi_{|a|}^0$ set R , we can find an index for a uniformly computable sequence $(\mathcal{C}_n)_{n \in \omega}$ such that

$$\mathcal{C}_n \cong \begin{cases} T_a & \text{if } n \in R \\ T_b & \text{otherwise} \end{cases}$$

For all n , \mathcal{A}_n will have infinitely many \sim -equivalence classes isomorphic to T , plus infinitely many \sim -equivalence classes isomorphic to T_a , for certain a . Suppose $|a| = \beta + 2$. If $n \notin S_\beta$, then there will be infinitely many \sim -equivalence classes on which the tree structure is isomorphic to T_a . Otherwise, there will be no \sim -equivalence classes on which the tree structure is isomorphic to T_a . In particular, if $|a|$ is not of the form $\beta + 2$, there will be no \sim -equivalence classes isomorphic to T_a .

Suppose for the moment that we can produce the sequence \mathcal{A}_n as described. If $n \in S$, then by Lemma 5.3, there exists γ such that for all $\beta \geq \gamma$, $n \in S_\beta$. Say $g \in P$ is the notation for $\gamma + 2$. Then for $a \geq g$ in P , there is no \sim -equivalence class isomorphic to T_a . For each $a < g$, let ψ_a be the effectively determined computable infinitary Scott sentence for T_a . Let ψ be the disjunction of these. The \sim -equivalence classes isomorphic to T are the ones that satisfy $neg(\psi)$. In this case, the Scott rank of \mathcal{A}_n is ω_1^{CK} .

If $n \notin S$, then by Lemma 5.3, there are arbitrarily large β such that $n \notin S_\beta$. If $a \in P$ is a notation for $\beta + 2$, where $n \notin S_\beta$, then there are infinitely many \sim -equivalence classes isomorphic to T_a . Any computable infinitary sentence true in T is also true in T_a for arbitrarily large a . Any computable infinitary formula true of the top node of a \sim -equivalence class isomorphic to T is also true of the top nodes of \sim -equivalence classes isomorphic to T_a for sufficiently large a . Therefore, \mathcal{A}_n has Scott rank $\omega_1^{CK} + 1$.

We must show that there exists a uniformly computable sequence of structures $(\mathcal{A}_n)_{n \in \omega}$ as described. We can write a Π_1^1 set Γ of computable infinitary sentences describing the desired sequence of structures \mathcal{A}_n . (We could put the sequence of structures together in a single structure, with a disjoint family of unary predicates for the universes of the separate structures \mathcal{A}_n .)

1. We say that \sim is an equivalence relation partitioning the universe of \mathcal{A}_n into infinitely many infinite classes.

2. Suppose $a \in P$, where $|a| = \beta + 2$.
- (a) If $n \notin S_\beta$, then we say that \mathcal{A}_n has infinitely many \sim -equivalence classes on which the tree structure is isomorphic to T_a . (We use the Scott sentence for T_a .)
 - (b) If $n \in S_\beta$, then we say that there is no \sim -equivalence class on which the tree structure is isomorphic to T_a . (Again we use the Scott sentence for T_a .)
 - (c) In either case, we say that \mathcal{A}_n has infinitely many \sim -equivalence classes on which the tree structure satisfies the computable $\Sigma_{|a|}$ sentences true in T , and each of the remaining \sim -equivalence classes is isomorphic to T_b for some $b < a$.

We want a *computable* model of Γ . This will yield a *uniformly computable* sequence of structures $(\mathcal{A}_n)_{n \in \omega}$ as described. We shall use the Barwise-Kreisel Compactness Theorem, or rather Corollary 2.4. Let Γ' be a Δ_1^1 subset of Γ . We may suppose that the ordinals involved in Γ' are all less than γ , and let g be the notation for γ in P . To show that Γ' has a computable model, we produce a uniformly computable sequence of structures $(\mathcal{A}'_n)_{n \in \omega}$. For each n , \mathcal{A}'_n will have infinitely many \sim -equivalence classes isomorphic to T_g . In addition, \mathcal{A}'_n will have infinitely many \sim -equivalence classes isomorphic to T_a if $a < g$, where $|a| = \beta + 2$ and $n \notin S_\beta$.

Let a be an element of P such that $|a| = \beta + 2$. The relation “ $n \notin S_\beta$ ” is $\Pi_{|a|}^0$ ($\Delta_{|a|}^0$ actually), uniformly in a . By Property III, we get a uniformly computable family of trees $T_{n,a}$, for $n \in \omega$ and $a < g$ such that a is a second predecessor, such that if $|a| = |b| + 2$, then

$$T_{n,a} \cong \begin{cases} T_a & \text{if } n \notin S_{|b|} \\ T_g & \text{otherwise} \end{cases}$$

Using the trees $T_{n,a}$, we obtain uniformly computable structures \mathcal{A}'_n , with infinitely many \sim -equivalence classes isomorphic to T_g and, for each $a < g$, infinitely many \sim -equivalence classes isomorphic to $T_{n,a}$. We have shown that Γ' has a computable model. Now, we are in a position to apply Corollary 2.4. We get uniformly computable structures \mathcal{A}_n as described in Γ . In the case where $(\exists x)[(n, x) \in S^1 - S^2]$, there is a bound on the $a \in P$ such that there are \sim -classes isomorphic to T_a . In the case where $(\forall x)[(n, x) \in S^1 - S^2]$, there is no bound on the $a \in P$ such that there are \sim -classes isomorphic to T_a .

Using Corollaries 2.5 and 2.6, we can show that there are infinitely many \sim -equivalence classes isomorphic to T . Let $\Lambda(x)$ say that x is the top node of a \sim -equivalence class satisfying the computable infinitary sentences true in T . Every Δ_1^1 subset is satisfied, so by Corollary 2.5, some a satisfies $\Lambda(x)$. By Corollary 2.6, the \sim -equivalence class of a is isomorphic to T . Supposing that \bar{a} consists of the top nodes in finitely many \sim -equivalence classes isomorphic to T , we get another one in the same way. Let $\Lambda(\bar{a}, x)$ say that x is the top node in

another \sim -equivalence class satisfying the computable infinitary sentences true in T . Again, every Δ_1^1 subset is satisfied, so the whole set is. \square

Now, we get the sharp result.

Theorem 5.5.

1. For all $S \in D_2$, $S \leq_m I(K_{\omega_1^{CK}})$ within $I(K)$.

2. For all $S \in I_2$, $S \leq_m I(K_{\omega_1^{CK+1}})$ within $I(K)$.

Proof. It is enough to prove 1. Suppose $S \in D_2$, with a definition of the form $(\forall m)(\exists x)R(n, m, x)$, where $R(n, m, x)$ is $\Pi_1^1 - \Pi_1^1$. Proceeding as in the proof of Theorem 5.2, we obtain a uniformly computable family of structures $\mathcal{A}_{n,m}$, for $n, m \in \omega$, such that

$$\begin{aligned} \mathcal{A}_{n,m} \in K_{\omega_1^{CK}} & \quad \text{if } (\exists x)R(n, m, x) \\ \mathcal{A}_{n,m} \in K_{\omega_1^{CK+1}} & \quad \text{if } (\forall x)R(n, m, x) \end{aligned}$$

For each n , let \mathcal{B}_n be the cardinal sum of the structures $\mathcal{A}_{n,m}$, with universe equal to the disjoint union of unary predicates U_m , where U_m is the universe of a copy of $\mathcal{A}_{n,m}$. It is not difficult to see that \mathcal{B}_n is in $K_{\omega_1^{CK}}$ if all $\mathcal{A}_{n,m}$ are in $K_{\omega_1^{CK}}$, and otherwise, it is in $K_{\omega_1^{CK+1}}$. \square

6 Conclusion

The work in the present paper was motivated in part by two questions, mentioned already in the introduction. These two questions remain open.

Question 1. *Suppose \mathcal{A} is a computable structure. If $I(\mathcal{A})$ is hyperarithmetical, must $SR(\mathcal{A})$ be computable?*

Question 2. *Is there a computable structure \mathcal{A} such that $SR(\mathcal{A})$ is non-computable and $I(\mathcal{A})$ is not m -complete Σ_1^1 ?*

We know more than we did when we began, but there is still a great deal that we do not understand about high rank structures.

References

- [1] Ash, C. J., and J. F. Knight, “Pairs of recursive structures”, *Annals of Pure and Appl. Logic*, vol. 46(1990), pp. 211–234.
- [2] Ash, C. J., and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, Elsevier, 2000.

- [3] Barwise, J., *Admissible Sets and Structures: An Approach to Definability Theory*, Springer, 1975.
- [4] Calvert, W., S. S. Goncharov, and J. F. Knight, “Computable structures of Scott rank ω_1^{CK} in familiar classes”, to appear in *Advances in Logic* (Proceedings of the North Texas Logic Conference, October, 2004), *Contemporary Mathematics*, American Mathematical Society.
- [5] Calvert, W., S. S. Goncharov, and J. F. Knight, “Boolean algebras of Scott rank ω_1^{CK} ”, preprint.
- [6] Calvert, W., V. S. Harizanov, J. F. Knight, and S. Miller, “Index sets for computable structures”, *Algebra and Logic*, vol. 45(2006), pp. 306–325.
- [7] Calvert, W., J. F. Knight, and J. Millar, “Trees of Scott rank ω_1^{CK} and computable approximability”, *Journal of Symbolic Logic* 71(2006) pp. 283–298.
- [8] Goncharov, S. S., and J. F. Knight, “Computable structure and non-structure theorems”, *Algebra and Logic*, vol. 41(2002), pp. 351–373.
- [9] Goncharov, S. S., V. S. Harizanov, J. F. Knight, and R. Shore, “ Π_1^1 relations and paths through \mathcal{O} ”, *J. Symb. Logic*, vol. 69(2004), pp. 585–611.
- [10] Keisler, H. J., *Model Theory for Infinitary Logic*, North-Holland, 1971.
- [11] Knight, J. F., and J. Millar, “Computable structures of rank ω_1^{CK} ”, submitted to *J. of Math. Logic*.
- [12] Knight, J. F., “A computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not \aleph_0 categorical”, in preparation.
- [13] Harrison, J., “Recursive pseudo well-orderings”, *Trans. of the Amer. Math. Soc.*, vol. 131(1968), pp. 526–543.
- [14] Makkai, M., “An example concerning Scott heights”, *J. Symb. Logic*, vol. 46(1981), pp. 301–318.
- [15] Millar, J., and G. Sacks, “Atomic models higher up”, preprint.
- [16] Nadel, M., “Scott sentences and admissible sets”, *Annals of Math. Logic*, vol. 7(1974), pp. 267–294.
- [17] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.
- [18] Sacks, G. E., *Higher Recursion Theory*, Springer-Verlag, 1990.
- [19] Scott, D., “Logic with denumerably long formulas and finite strings of quantifiers”, in *The Theory of Models*, ed. by J. Addison, L. Henkin, and A. Tarski, North-Holland, 1965, pp. 329–341.

- [20] W. White, “On the complexity of categoricity in computable structures”
Math. Logic Quarterly, vol. 49 (2003), pp. 603–614.
- [21] W. White, *Characterizations for Computable Structures*, PhD dissertation,
Cornell University, 2000.