

High Scott rank and the categoricity of computable infinitary theories

W. Calvert, S. S. Goncharov, J. F. Knight, and J. Millar

May 19, 2007

Makkai [7] gave an example of an arithmetical structure of Scott rank ω_1^{CK} . In [6], the example is made computable. In [3], there is a computable tree of Scott rank ω_1^{CK} . In [1], [2], the tree is used to produce further structures of Scott rank ω_1^{CK} in the following classes: undirected graphs, fields of any desired characteristic, linear orderings, and Boolean algebras. The technique is to consider examine effective transformations from one class to another and show that they have a rank preservation property.

For a structure \mathcal{A} , let $T_\infty(\mathcal{A})$ be the computable infinitary theory of \mathcal{A} . For Makkai's example [7] and the computable variant constructed in [6], the computable infinitary theory is \aleph_0 -categorical. There are other examples of computable structures of Scott rank ω_1^{CK} . In [3], there is a tree. In [1] and [2], certain rank-preserving effective embeddings are used to transform the tree into an undirected graph, a field (of any desired characteristic), a linear ordering, and a Boolean algebra. In the remainder of the present section, we will state the important definitions. In Section 2, we state a general result which is sufficient to imply categoricity. In Section 3, we apply this result to the examples produced in [3] and [1].

There are different definitions of Scott rank in use. Instead of choosing one, we give the following useful result. We write ω_1^X for the first ordinal not computable in the set X .

Proposition 1. *Suppose $X \equiv_T \mathcal{A}$.*

1. $SR(\mathcal{A}) < \omega_1^X$ if there is some $\alpha < \omega_1^X$ such that the orbits of all tuples are definable by X -computable Π_α formulas,
2. $SR(\mathcal{A}) = \omega_1^X + 1$ if there is some tuple whose orbit is not definable by any X -computable formula,
3. $SR(\mathcal{A}) = \omega_1^X$ if the orbits of all tuples are defined by X -computable formulas, but there is no bound $\alpha < \omega_1^X$ as in 1.

Our definition of rank preserving embedding calls for preserving the values that matter.

Definition 1. A transformation Φ from K to K' has the rank preservation property if for $\mathcal{A} \in K$ and $\Phi(\mathcal{A}) = \mathcal{B}$, either $SR(\mathcal{A}) = SR(\mathcal{B})$ or else both Scott ranks are computable relative to \mathcal{A} .

1 A General Result

Theorem 2. Let \mathcal{A} be a computable structure. Let $(\delta_n(x))_{n \in \omega}$ be a computable sequence of computable infinitary formulas of bounded complexity (computable Σ_α for some computable ordinal α). Suppose that each element of \mathcal{A} satisfies one of the formulas $\delta_n(x)$. Finally, suppose that for each n , there is a computable bound on the Scott rank of tuples \bar{a} such that each element of \bar{a} satisfies δ_m , for some $m \leq n$. Then $T_\infty(\mathcal{A})$ is \aleph_0 -categorical.

Proof. We write $\delta_{\leq n}$ for the formula saying of a tuple of variables that each element of the tuple satisfies δ_m for some $m \leq n$. For each n , the fact that there is a computable bound on the Scott ranks of tuples satisfying $\delta_{\leq n}$ means that there is a computable ordinal α_n such that for tuples \bar{u}, \bar{v} satisfying $\delta_{\leq n}$, if \bar{u} and \bar{v} satisfy the same computable Σ_{α_n} formulas, then they are automorphic. For each tuple \bar{a} , let $\Gamma_{\bar{a}, n}(\bar{u})$ be the set of computable Σ_{α_n} formulas satisfied by \bar{a} .

The theory $T_\infty(\mathcal{A})$ includes sentences saying the following:

1. each element satisfies some $\delta_n(x)$
2. if \bar{u} satisfies $\delta_{\leq n}$, then it satisfies one of the sets $\Gamma_{\bar{a}, n}$, where \bar{a} satisfies $\delta_{\leq n}$ in \mathcal{A} ,
3. there is some \bar{u} satisfying $\delta_{\leq n}$ and $\Gamma_{\bar{a}, n}$, for each \bar{a} satisfying $\delta_{\leq n}$ in \mathcal{A} .
4. if \bar{u} and \bar{v} satisfy $\delta_{\leq n}$, and they satisfy the same computable Σ_{α_n} formulas, then they also satisfy the same computable Σ_{n+1} formulas.

Suppose \mathcal{B} is a countable model of $T_\infty(\mathcal{A})$. We can show that $\mathcal{A} \cong \mathcal{B}$. Let Φ be the set of formulas saying, for some n , that \bar{u} satisfies $\delta_{\leq n}(\bar{u})$ and also satisfies $\Gamma_{\bar{a}, n}(\bar{u})$, where \bar{a} is a tuple satisfying $\delta_{\leq n}$ in \mathcal{A} . This is a Scott family for \mathcal{A} . We refer to n as the level of the formula. We show that the set of finite partial 1 – 1 functions mapping \bar{a} to \bar{b} satisfying the same formula in Φ has the back-and-forth property. Say \bar{a} satisfies a formula $\varphi(\bar{u})$ at level m in Φ . For any c , take $n \geq m$ such that \bar{a}, c satisfies a formula $\psi(\bar{u}, v)$ at level n in Φ . By 4 above, for any tuple satisfying $\varphi(\bar{u})$, there is some v satisfying $\psi(\bar{u}, v)$. Therefore, there exists d such that we can map \bar{a}, c to \bar{b}, d . □

2 Examples: Trees, Graphs, Fields, Linear Orderings

For the computable tree \mathcal{T} in [3] of Scott rank ω_1^{CK} , the computable infinitary theory is \aleph_0 -categorical. The tree is *rank homogeneous*.

Definition 2. *Suppose \mathcal{T} is a subtree of $\omega^{<\omega}$. Let \mathcal{T}_n denote the set of elements at level n (i.e., of length n). We say that \mathcal{T} is rank homogeneous if for all $x \in \mathcal{T}_n$,*

1. *if $tr(x) = \alpha$, and there exists $y \in \mathcal{T}_{n+1}$ such that $tr(y) = \beta$, where $\beta < \alpha$, then x has infinitely many successors of rank β ,*
2. *if $tr(x) = \infty$, then x has infinitely many successors y of rank ∞ .*

A computable rank homogeneous tree has Scott rank ω_1^{CK} if and only if for each n , there is a computable bound α_n on the ordinals that occur as ranks of elements at level n , but there is no computable bound over all (see [3]).

Proposition 3. *Let \mathcal{T} be a computable rank homogeneous tree of Scott rank ω_1^{CK} . Then $T_\infty(\mathcal{T})$ is \aleph_0 categorical.*

Proof. We apply the theorem above. letting $\delta_n(x)$ say that x is at level n in the tree. Now the complexity of the formulas δ_n are bounded, and all elements are at some level. Since the tree is of Scott rank ω_1^{CK} , the ranks of elements at each level are bounded. Now the hypotheses of Theorem 2 are satisfied, and the result holds. □

To produce a computable undirected graph of Scott rank ω_1^{CK} , we use a familiar coding method, described, for example in Marker [8], p. ?. We can pass effectively from a tree \mathcal{T} to an undirected graph \mathcal{G} such that in \mathcal{G} , we can define, in a uniform way, a copy $\mathcal{T}^{\mathcal{G}}$ of \mathcal{T} , and if D is the universe of $\mathcal{T}^{\mathcal{G}}$, then the orbits of tuples in \mathcal{G} over D are all defined by finitary existential formulas.

Proposition 4. *Let \mathcal{G} be the undirected graph corresponding to a computable rank-homogeneous tree \mathcal{T} of Scott rank ω_1^{CK} . Then $T_\infty(\mathcal{G})$ is \aleph_0 -categorical.*

Proof. Let $\delta_n(x)$ be as in Proposition 3, and let $\delta_n^*(x)$ be the natural formula saying that x is in the subgraph corresponding to the subtree consisting of elements satisfying δ_n (i.e. tree elements at level n). Now the complexities of the formulas δ_n^* are bounded because those of the formulas δ_n are. By the definition of the graph, every element will satisfy some δ_n^* .

Suppose that α is a bound on the Scott ranks of the tuples whose members all satisfy δ_n . Let \bar{a} be a tuple from \mathcal{G} whose members all satisfy δ_n^* , and let I be the smallest set such that \bar{a} is in the subgraph corresponding to the subtree consisting of elements satisfying δ_n . Now I will be finite, and all elements of I will satisfy δ_n , so that $SR(I) \leq \alpha$. Now there will be a bound on the ranks of

elements of \bar{a} which is computable from the ranks of elements of I . Thus, the hypotheses of Theorem 2 are satisfied, and the result follows. \square

For any characteristic p , there is an effective procedure which, given a graph \mathcal{G} , will produce a field \mathcal{F} of characteristic p , in a way that respects isomorphism and Scott rank ([1], using a construction from [4]). For the case of characteristic zero, we let F_0 be the smallest field containing \mathbb{Q} and an uniformly computable set $(X_i)_{i \in \omega}$ of algebraically independent elements. The field \mathcal{F} is

$$F_0(\{\sqrt{a+b} : (a, b) \in G^s\},$$

where G^s is the set of pairs (a, b) such that a is interalgebraic with X_i and b is interalgebraic with X_j , with i and j connected in \mathcal{G} . There is an interpretation of \mathcal{G} in \mathcal{F} , and the orbits of tuples in \mathcal{F} are defined by finitary existential formulas with parameters in the set D whose equivalence classes form the universe of the interpretation of \mathcal{G} .

Proposition 5. *Let \mathcal{F} be as described above. Then $T_\infty(\mathcal{F})$ is \aleph_0 categorical.*

Proof. Let δ_n be the formula from Proposition 4. Let δ_n^* say that x is algebraic over the set of X_i such that $\delta_n(i)$ is true in \mathcal{G} . There is an effective procedure to pass from the complexity of δ_n to the complexity of δ_n^* , and so the complexities of the formulas δ_n^* are bounded. Since the entire field \mathcal{F} is algebraic over the set of all X_i , the finite character of algebraic closure guarantees that every element satisfies some δ_n^* .

Suppose that α is a bound on the Scott ranks of the tuples whose members all satisfy δ_n . Let \bar{a} be a tuple from \mathcal{F} whose members all satisfy δ_n^* , and let I be the smallest set such that \bar{a} is algebraic over $\{X_i : i \in I\}$. Now I will be finite, and all elements of I will satisfy δ_n , so that $SR(I) \leq \alpha$. Now there will be a bound on the ranks of elements of \bar{a} which is computable from the ranks of elements of I . Thus, the hypotheses of Theorem 2 are satisfied, and the result follows. \square

From the graph \mathcal{G} , we obtain a linear ordering \mathcal{L} in [1], using a construction from [4]. We start with the lexicographic ordering on $\mathbb{Q}^{<\omega}$. Let $(t_n)_{n \in \omega}$ be a uniformly computable list of the atomic types for tuples in graphs, such that the types with m variables appear before those with $m + 1$ variables. Let $(Q_a)_{a \in \omega}$ be a computable partition of \mathbb{Q} into dense subsets. The sets Q_0 and Q_1 will have special roles. The ordering \mathcal{L} will be the subordering of $\mathbb{Q}^{<\omega}$ consisting of the sequences of the form $r_0 q_1 r_1 q_2 r_2 \dots q_n r_n k$ such that for some $a_1, \dots, a_n \in \mathcal{G}$, of atomic type t_m , we have $q_i \in Q_{a_i}$ for all $i \leq n$, and such that $r_n \in Q_1$ and $k < m$. Thus, the largest discrete interval containing such a sequence will tell the atomic type of the sequence \bar{a} in \mathcal{G} .

Proposition 6. *Let \mathcal{L} be the ordering described above. Then $T_\infty(\mathcal{L})$ is \aleph_0 -categorical.*

Proof. We will apply Theorem 2 again. Let δ_n be the formulas associated with the graph. There is a notion of *level* in \mathcal{L} , where the elements at level ℓ belong to finite discrete orderings which represent tuples in \mathcal{G} of length ℓ . We have a notion of successor—meaning that the tuple is extended by one element. For each j and ℓ , we can define the set of elements at level ℓ representing tuples satisfying $\delta_{\leq j}$. Moreover, the formulas involved have bounded complexity. We let $\delta_{j,\ell}^*(x)$ say that x is in this set. We get a sub-ordering \mathcal{L}_j , associated with the subgraph consisting of elements satisfying $\delta_{\leq j}$ —this is the union of the sets satisfying $\delta_{j,\ell}^*$. Let δ_j^* be the disjunction of the formulas $\delta_{j,\ell}^*$.

Now the formulas δ_j^* have bounded complexity, and each element of \mathcal{L} satisfies one of the formulas. We will now show that the Scott ranks of tuples satisfying each of the δ_j^* are bounded. Fix j , and let α be the bound on the Scott ranks of tuples in \mathcal{G} satisfying $\delta_{\leq j}$. Let a satisfy δ_j^* . We showed in Claim 2 toward Corollary 3.6 of [1] that there is a computable function g such that if a is a sequence ending in 0, we have $g(a) \in \mathcal{G}$, and for all $b \in \mathcal{L}$ which end in 0, we have $b \simeq a$ if and only if $g(b) \simeq g(a)$ in \mathcal{G} . Thus, if a ends in 0, then a has Scott rank which is computable from α in a way which is uniform in a . In the case that a ends in $k \neq 0$, the orbit of a is determined by k and by the orbit of the least element in the longest discrete interval containing a , and so the Scott ranks of elements automorphic to a are bounded in this case, as well. Now the hypotheses of Theorem 2 are satisfied, and the conclusion follows. \square

References

- [1] Calvert, W., S. S. Goncharov, and J. F. Knight, “Computable structures of Scott rank ω_1^{CK} in familiar classes”, to appear in *Contemporary Mathematics*, proceedings of North Texas meeting, ed. by Gao, Jackson, and Zhang.
- [2] Calvert, W., S. S. Goncharov, and J. F. Knight, “Boolean algebras and rank preservation”, pre-print.
- [3] Calvert, W., J. F. Knight, and J. Millar, “Trees of Scott rank ω_1^{CK} , and computable approximability”, *J. Symb. Logic*, vol. 71 (2006), pp. 283–298.
- [4] Friedman, H., and L. Stanley, “A Borel reducibility theory for classes of countable structures”, *J. Symb. Logic*, vol. 54(1989), pp. 894–914.
- [5] Hirschfeldt, D., B. Khoushainov, A. Slinko, and R. Shore, “Degree spectra and computable dimension in algebraic structures”, *Annals of Pure and Appl. Logic*, vol. 115(2002), pp. 71–113.
- [6] Knight, J. F., and J. Millar, “Computable structures of Scott rank ω_1^{CK} ”, submitted to *J. Math. Logic*.
- [7] Makkai, M., “An example concerning Scott heights”, *J. Symb. Logic*, vol. 46(1981), pp. 301–318.

[8] Marker, D., *Model theory: An Introduction*, Springer-Verlag, 2002.