Geometry of Definable Sets over non-Archimedean Fields

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1 Definable Sets

Let $\mathcal{L}$ be a language and $\mathcal{A}$ be an $\mathcal{L}$-structure with domain A. We say that a subset $S \subset A^n$ is *$\mathcal{L}$-definable* if there is a formula ϕ in the language $\mathcal{L}$ such that

$$S = \{\bar{p} \in A^n : \phi(\bar{p})\}.$$

If we can also choose ϕ to be a quantifier-free formula, then S is said to be *quantifier-free definable.*

Examples:

- Any finite or cofinite subset of $\mathbb{Q}$ is quantifier-free definable in the language $\mathcal{L}_{alg} = \{0, 1, +, -, \cdot\}$.
- Conversely, any quantifier-free definable subset of a field K in $\mathcal{L}_{alg}$ is either finite or cofinite.

Why?

-Every $\mathcal{L}_{alg}$-term is a polynomial with integer coefficients.

-Put ϕ into disjunctive normal form:

$\bigvee_i \bigwedge_j (p_{ij}(x) \Box 0)$ where $\Box$ is either $=$ or $\neq$.

- In general, for a field K, any $\mathcal{L}_{alg}$-quantifier-free definable subset of K^n is a boolean combination of algebraic varieties.
- Moreover if K is algebraically closed, then any $\mathcal{L}_{alg}$*-definable* subset of K^n is a boolean combination of algebraic varieties.

2 Quantifier Elimination

Let $\mathcal{L}$ be a language and T be a theory of $\mathcal{L}$. T *admits elimination of quantifiers in* $\mathcal{L}$ if for every $\mathcal{L}$-formula ϕ there is a quantifier free $\mathcal{L}$-formula ψ such that

$$T \vdash \phi \Leftrightarrow \psi.$$

This is equivalent to the class of quantifier free definable sets being closed under projection.

Theorem: (Chevalley's Theorem) The theory of algebraically closed fields in the language $\mathcal{L}_{alg}$ admits elimination of quantifiers.

Corollary: The theory of algebraically closed fields of a given characteristic in $\mathcal{L}_{alg}$ is complete.

Why?

-Every sentence in $\mathcal{L}$ is equivalent to a quantifier free sentence ψ.

-$\psi = \bigvee_i \bigwedge_j (t_{ij} \square 0)$ where $\square$ is $=$ or $\neq$

-Each t_{ij} is a polynomial without variables, hence an integer.

-ψ is a statement about the characteristic of the field.

-The theory of real numbers in the language $\mathcal{L}_{alg}$ does not admit elimination of quantifiers.

Why? -The set $P_2 = \{x \in \mathbb{R} : \exists y(y^2 = x)\}$ is neither finite nor cofinite.

Theorem: (Tarski's Theorem) The theory of real closed fields admits elimination of quantifiers in the language

$$\mathcal{L} = \{0, 1, +, -, \cdot, <\}.$$

3 Non-Archimedean Valued Fields

Let A be a ring. A map $|\ |: A \to \mathbb{R}$ is called a Non-Archimedean Valuation (norm) on A if

i) $|0| = 0$, and $|x| > 0$ for all $x \neq 0$,

ii) $|x+y| \leq \max\{|x|, |y|\}$,

iii) $|xy| = |x||y|$.

Note that $|A|$, the image set of elements in A under the valuation map, is a multiplicative semi-group.

Examples:

• Fix a prime number p and define the norm $|\ |_p$ on $\mathbb{Z}$ as follows

$$\begin{aligned} |0|_p &= 0 \\ |n|_p &= p^{-k} \quad \text{if } p^k | n \text{ and } p^{k+1} \not| n \text{ for } n \neq 0. \end{aligned}$$

- Completion of $\mathbb{Z}$ with respect to the norm $|\ |_p$ is the ring of p-adic integers $\mathbb{Z}_p$, the quotient field $\mathbb{Q}_p$ of $\mathbb{Z}_p$ is the field of p-adic numbers. Every element a of $\mathbb{Q}_p$ can be written in the form

$$a = a_n p^n + a_{n+1} p^{n+1} + \ldots$$

for some $n \in \mathbb{Z}$ and $0 \leq a_i < p$ for all i. The operations $+$ and $\cdot$ work the obvious way.

• Let K be a field, then the field $K((t))$ is a non-Archimedean field with the t-adic valuation

$$\begin{aligned} |0|_t &= 0 \\ \left| \frac{a_k t^k + a_{k+1} t^{k+1} + \ldots}{b_l t^l + b_{l+1} t^{l+1} + \ldots} \right|_t &= 2^{l-k} \text{ for } a_k \neq 0, b_l \neq 0. \end{aligned}$$

Theorem: (Macintyre's Theorem) The theory of $\mathbb{Z}_p$ admits elimination of quantifiers in the language

$$\mathcal{L} = \{0, 1, +, -, \cdot, P_2, P_3, ...\}$$

where

$$P_i(x) \Leftrightarrow \exists y (y^i = x).$$

What happens if we add "analytic functions" as terms in our language?

- For a non-Archimedean field K, the metric topology on a non-Archimedean Valued Field K^n is totally disconnected, a function f defined as

$$f = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| \geq 1 \end{cases}$$

is locally given by convergent power series. So the usual notion of analytic functions is insufficient.

- From now on K will denote a complete non-Archimedean field. (**Example:** $\mathbb{Q}_p$).

- A series $\sum_i a_i$ where $a_i \in K$ converges if and only if $|a_i| \to 0$.

- We will write K° for the closed unit ball $\{a \in K : |a| \leq 1\}$ and $K^{\circ\circ}$ for $\{a \in K : |a| < 1\}$.

- *Rigid analytic functions* over $(K^\circ)^n$ are elements of $K\langle x_1, ..., x_n\rangle$, strictly convergent power series ring over K. That is, the ring of power series over n variables $\sum_\alpha a_\alpha x^\alpha$ with the property $|a_\alpha| \to 0$ as $|\alpha| \to \infty$. In other words, they are exactly the power series that converge on the closed unit ball K°.

- $K\langle x_1, ..., x_n\rangle$ is also called the Tate Algebra over K.

Theorem: (Denef, van den Dries) The theory of $\mathbb{Z}_p$ admits elimination of quantifiers in the language

$$\mathcal{L}_{an}^{D} = \{+, -, \cdot, P_2, P_3, ..., \mathbb{Z}_p, \mathbb{Z}_p \langle x_1 \rangle, \mathbb{Z}_p \langle x_1, x_2 \rangle, ..., D\}.$$

where $D : \mathbb{Z}^2 \to \mathbb{Z}^2$ is the restricted division operation

$$D(x, y) = \begin{cases} x/y & \text{if } |x| \leq |y| \text{ and } y \neq 0 \\ 0 & \text{if } |x| > |y| \text{ or } y = 0. \end{cases}$$

4 The Proof of Analytic Quantifier Elimination

- An element $a = \sum a_\alpha x^\alpha$ of $K \langle x_1, ..., x_n \rangle$ is said to be regular in x_1 of degree s if $|a_{(s,0,\ldots,0)}| \geq |a_\alpha|$ for all $\alpha <_{lex} (s, 0, ..., 0)$ and if $|a_{(s,0,\ldots,0)}| > |a_\beta|$ for all $\beta >_{lex} (s, 0, ..., 0)$.

Examples:

- $f(x, y) = p^2y^3 + px^3 + p^2xy^2 + py^2 + p^3 \in \mathbb{Q}_p \langle x, y \rangle$

is regular of degree 2 in y and of degree 3 in x.

- p^2xy^2 is not regular in any variable in any degree.

Theorem: (Weiertrass Preparation Theorem) If $f \in K \langle x_1, ..., x_n \rangle$ is regular of degree s in x_1, then there exists a unit $u \in K \langle x_1, ..., x_n \rangle$and a monic polynomial $g \in K \langle x_2, ..., x_n \rangle [x_1]$ which is regular of degree s in x_1 such that

$$f = u \cdot g.$$

Outline:

- The relation $|x| \leq |y|$ is quantifier free definable in $\mathcal{L}_{an}^{D}$.

- For a quantifier free $\mathcal{L}_{an}^{D}$-formula $\phi(x_1, ..., x_m)$ there is a quantifier free $\mathcal{L}_{an} = \mathcal{L}_{an}^{D} \setminus \{D\}$-formula $\bar{\phi}(x_1, ..., x_m, z_1, ..., z_k)$ such that

$$\phi \Leftrightarrow \exists z_1, ..., z_k \bar{\phi}.$$

Replace the term $D(t_1, t_2)$ with z_1 and add the condition

$$(|t_1| \leq |t_2| \wedge t_2 \neq 0 \wedge t_1 = t_2 z_1)$$

$$\vee((|t_1| > |t_2| \vee t_2 = 0) \wedge z_1 = 0).$$

- A subset of $(K^{\circ})^m$ defined by a quantifier-free $\mathcal{L}_{an}^{D}$ formula is called *D-semianalytic.*

- It is enough to show that for a $\exists y \phi(x, y)$ where ϕ is a quantifier free $\mathcal{L}_{an}^{D}$-formula, we can find a quantifier free $\mathcal{L}_{an}^{D}$-formula $\psi(x)$ such that

$$\exists y \phi(x, y) \Leftrightarrow \psi(x).$$

$(x = x_1, ..., x_m)$, $y = (y_1, ..., y_n))$

- Construct the $\mathcal{L}_{an}$-formula $\bar{\phi}(x, z, y)$. Let $f_i = \sum_{\alpha} a_{i,\alpha}(x, z) y^{\alpha} \in K \langle x, z, y \rangle$ be the terms appearing in $\bar{\phi}$.

- There are finitely many candidates of $a_{i\alpha}(x, z)$ for being the largest coefficient and

$$A_{i\alpha} = \{(x, z) \in (K^{\circ})^{m+k} : a_{i\alpha}(x, z) \text{ is the largest coefficient}\}$$

is a D-semianalytic set.

- Over $A_{i\alpha}$, after a Weierstrass map $\varphi : y_n \to y_n,\ y_i \to y_i + y_n^{r_i}$ for $i < n$, $D(f_i, a_{i,\alpha})$ becomes regular in y_n.

- By the Weierstrass Preparation Theorem, the problem reduces to Macintyre's Theorem.

5 Rings of Separated Power Series

- The value group $\{..., p^2, p, 1, p^{-1}, p^{-2}, ...\}$ of $\mathbb{Q}_p \setminus \{0\}$ is discrete. Hence a relation of the type $|t_1| < |t_2|$ is easily seen to be expressible in $\mathcal{L}_{an}^D$ as $|t_1| \leq |p \cdot t_2|$.

- The proof of analytic quantifier elimination for $\mathbb{Z}_p$ does not extend to algebraic closure of $\mathbb{Z}_p$ as the value group is not discrete.

- Lipshitz introduced the ring of separated power series $S_{m,n}$ to work with an algebraically closed complete non-Archimedean field K.

- There are two types of variables: $x_1, ..., x_m$ ranging over K° and $\rho_1, ..., \rho_n$ ranging over $K^{\circ\circ}$.

- The members of $S_{m,n}$ converge on $(K^\circ)^m \times (K^{\circ\circ})^n$.

Theorem: (Lipshitz) The theory of an algebraically closed complete non-Archimedean field K admits elimination of quantifiers in the language

$$\mathcal{L}_{sep}^D = \{+, -, \cdot, K^\circ, |K^\circ|, <, S_{1,0}, S_{0,1}, S_{1,1}, ..., D_0, D_1\},$$

where D_0 and D_1 are restricted division operations

$$D_0(x,y) \quad = \quad \begin{cases} x/y & \text{if } |x| \leq |y| \text{ and } y \neq 0 \\ 0 & \text{if } |x| > |y| \text{ or } y = 0. \end{cases}$$

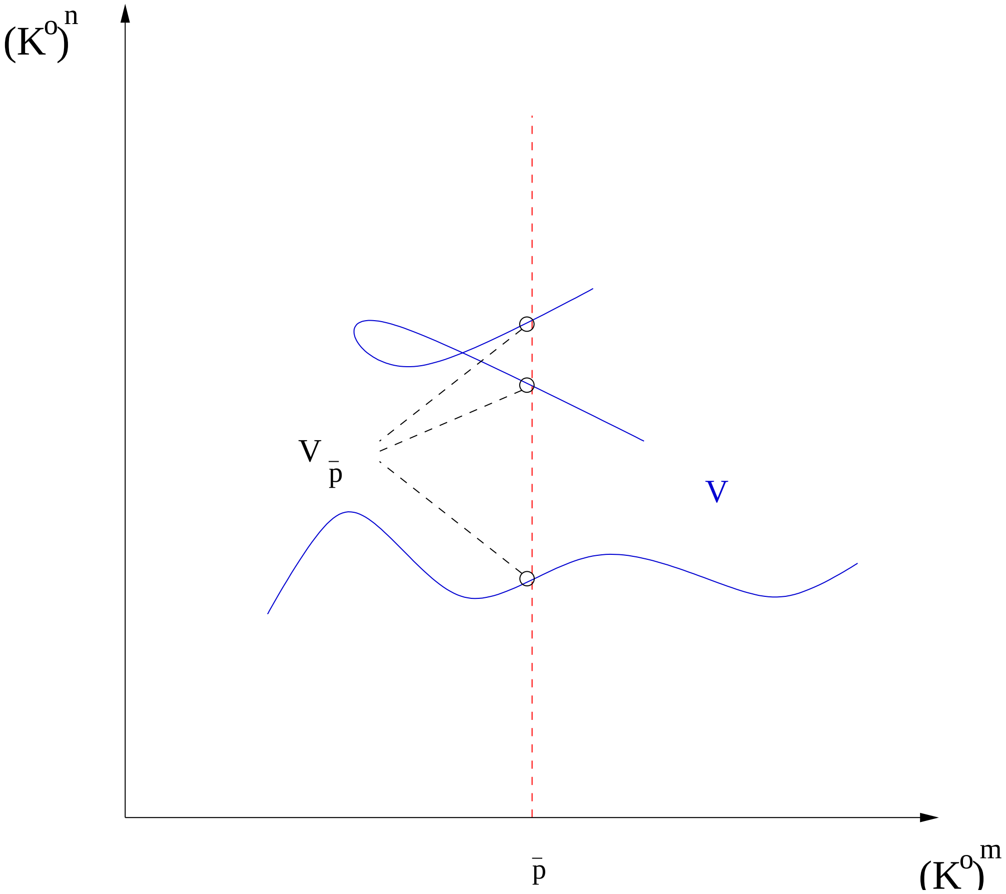
$$D_1(x,y) \quad = \quad \begin{cases} x/y & \text{if } |x| < |y| \\ 0 & \text{if } |x| \geq |y|. \end{cases}$$

6 Geometry of Definable Subsets of non-Archimedean fields

Theorem: (Lipshitz) Assume either char(K)=0 or if Char $K = p$, then $[K : K^p] < \infty$. Let $f_1, ..., f_k \in K\langle x_1, ..., x_m, y_1, ..., y_n\rangle$, and let V be the variety $V(f_1, ..., f_k)$ then there is a bound $N \in \mathbb{N}$ such that for each $\bar{p} \in (K^\circ)^m$, the fiber

$$V_{\bar{p}} = \{y \in (K^\circ)^n : f_i(\bar{p}, y) = 0 \text{ for all } i\}$$

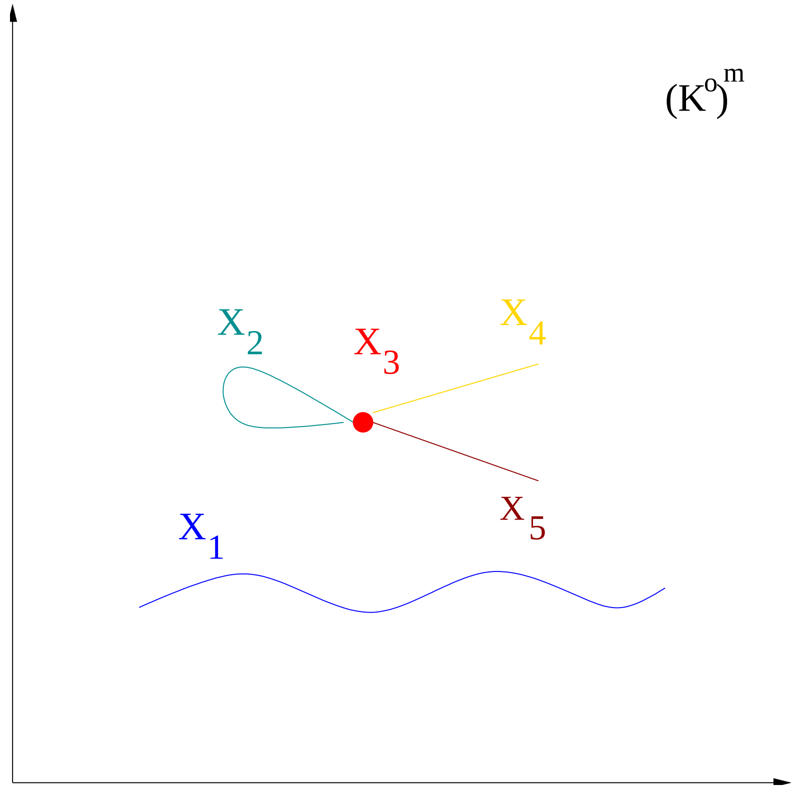
contains at most N isolated points.



Theorem: (Lipshitz, Robinson) Let K be an algebraically closed field, X be a definable subset of K^m in $\mathcal{L}_{sep}^D$ with dim $X = d$, then there are $\mathcal{L}_{sep}^D$-definable K-analytic manifolds $X_1, ..., X_l$ such that

$$X = X_1 \cup ... \cup X_l,$$

and $\max_i$ dim $X_i = d$.



- An $\mathcal{L}_{sep}^D$-definable subset of $(K^\circ)^m$ is called subanalytic, an $\mathcal{L}_{sep}^D$-quantifier free definable set is called D-semianalytic. These two classes of subsets coincide when K is algebraically closed.

7 The Domain of a Generalized Ring of Fractions

- We are interested in $\mathcal{L}_{sep}^{D}$-quantifier free definable subsets of $(K^{\circ})^m \times (K^{\circ\circ})^n$, where K is not necessarily algebraically closed.

- Let $S = \{\bar{p} \in (K^{\circ})^m \times (K^{\circ\circ})^n : \phi(\bar{p})\}$, ϕ quantifier-free. Write

$$\phi = \bigvee_i \bigwedge_j (|f_{ij}| \Box |g_{ij}| \wedge h_{ij} = 0).$$

($\Box$ is $<$ or $\leq$).

- By introducing new variables $x_{m+1}, ..., x_{m+s}$ and $\rho_{n+1}, ..., \rho_{n+t}$ and extra conditions, we may work with an $\mathcal{L}_{sep}$-formula ψ and take the projection of

$$S' = \{\bar{p} \in (K^{\circ})^{m+s} \times (K^{\circ\circ})^{n+t} : \bar{\phi}(\bar{p})\}$$

onto $(K^{\circ})^m \times (K^{\circ\circ})^n$ to get S.

- For an $\mathcal{L}_{sep}$-formula of the form

$$\psi = \bigwedge_i (|f_i| \leq |g_i|) \wedge \bigwedge_j (|F_j| < |G_j|).$$

we associate the ring

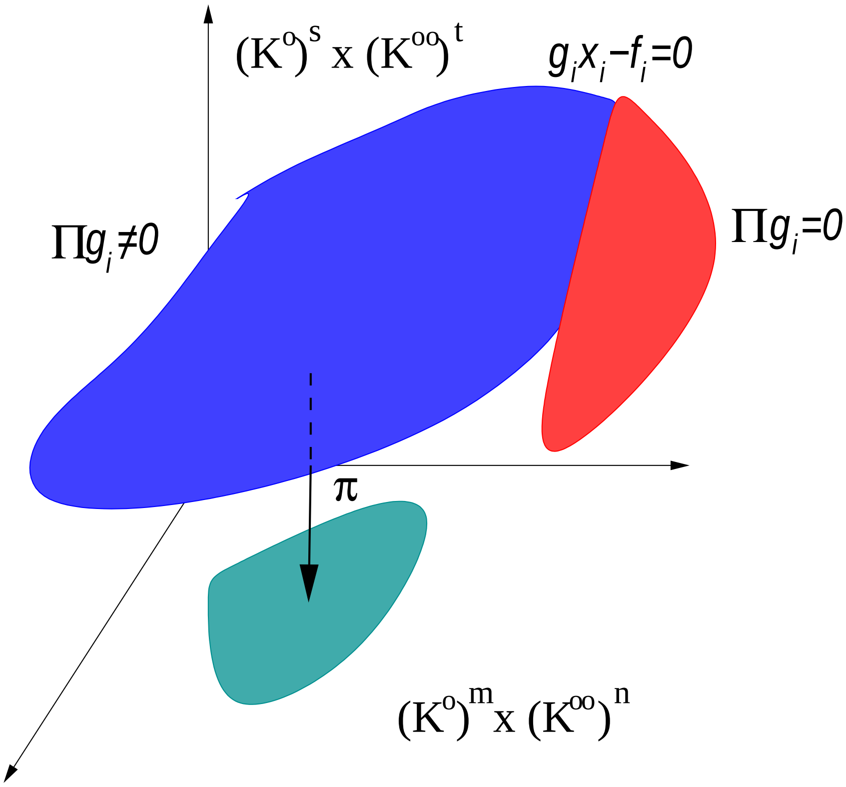
$$\begin{aligned} A &:= S_{m,n} \left\langle \left\{ \frac{f_i}{g_i} \right\}_i \right\rangle \left[\left[\left\{ \frac{F_j}{G_j} \right\}_j \right]\right]_s \\ A &= S_{m,n} \langle x_{m+1}, ..., x_{m+s} \rangle [[\rho_{n+1}, ..., \rho_{n+s}]]_s \,/ \\ &\qquad (\{g_i x_{m+i} - f_i\}_i \cup \{G_j \rho_{n+j} - F_j\}_j) \end{aligned}$$

- Define the Domain of the generalized ring of fractions A as the projection of the set

$$\{(x,\rho)\in (K^\circ)^{m+s}\times (K^{\circ\circ})^{n+t} : g_i x_{m+i} - f_i = 0, G_j\rho_{n+j} - F_j = 0, \prod g_i \cdot \prod G_j \neq 0\}.$$

onto $(K^\circ)^m \times (K^{\circ\circ})^n$.

- Then any D-semianalytic set is a union of sets of the form $\mathrm{Dom}(A_i)\cap V(I_i)$ where I_i is an ideal of A_i.



8 Dimension Theory

Definition: For a non-empty subset X of K^m, define the geometric dimension g-dim X to be the greatest integer d such that the image of X under a coordinate projection to a d-dimensional coordinate space has an interior point.

Define the weak dimension the same with the interior point condition replaced with being somewhere dense.

- w-dim $X \geq$ g-dim X.

- Also for a D-semianalytic set of the form $\mathrm{Dom}(A) \cap V(I)$ we have the Krull dimension of A/I.

- Define the restricted dimension r-dim of A/I to be the maximum length d of a prime ideal chain

$$I \subset \mathfrak{p}_0 \subset ... \subset \mathfrak{p}_d \subset A$$

where the point $\bar{p} \in (K^{\circ})^{m+s} \times (K^{\circ\circ})^{n+t}$ corresponding to $\mathfrak{p}_d$ satisfies $\prod g_i(\bar{p}) \cdot \prod G_j(\bar{p}) \neq 0$.

Theorem: Let Char $K = 0$ and X be $\mathrm{Dom}(A) \cap V(I)$ with $I = \mathcal{I}(\mathrm{Dom}(A) \cap V(I))$ and r-dim $A/I = d$, then there are D-semianalytic K-analytic manifolds $X_1, ..., X_l$ such that

$$X = X_1 \cup ... \cup X_l$$

$\max_i$ g-dim $X_i \geq d$.

9 Parameterized Normalization

- There is no "global" normalization for an algebra of the form $S_{m,n}/I$.

That is we can't find an $S_{m',n'}$ and a homomorphism φ such that

$$\varphi : S_{m',n'} \to S_{m,n}/I$$

is a finite injection.

- We will write $S_{m+M,n+N}$ for the separated power series ring over the variables $x_1, ..., x_m$, $\rho_1, ..., \rho_n$ (parameter variables), $y_1, ..., y_M$, and $\lambda_1, ..., \lambda_N$.

Theorem: Let B be a *grf* over $S_{m+M,n+N}$, then there are grfs B_i extending B and ideals $I_i \subset B_i$, rings $S_{m_i+M_i,n_i+N_i}$, and Weierstrass maps ϕ_i among x, ρ, y, λ variables separately such that

i) $\mathrm{Dom}(B) \cap V(I) = \bigcup_i \mathrm{Dom}(B_i) \cap V(I_i)$,

ii) $\phi_i : S_{m_i+M_i,n_i+N_i} \to B_i/I_i$ is a finite injection for each i.

Corollary: Let X be $\mathrm{Dom}(A) \cap V(I)$ with $I = \mathcal{I}(\mathrm{Dom}(A) \cap V(I))$ then r-dim $A/I \geq$ w-dim X.

Corollary: Let Char $K = 0$, then for $X = \mathrm{Dom}(A) \cap V(I)$, $I = \mathcal{I}(\mathrm{Dom}(A) \cap V(I))$ we have

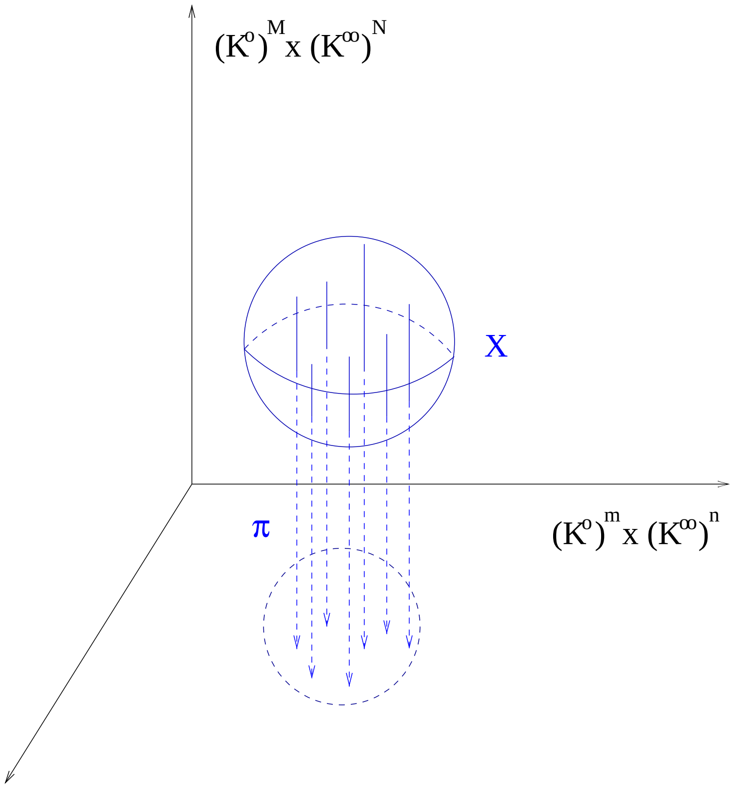
$$\text{r-dim } A/I = \text{ g-dim } X = \text{ w-dim } X.$$

Theorem: Let Char $K = 0$ and $X \subset (K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N}$ be a D-semianalytic set, and π be the coordinate projection

$$(K^\circ)^{m+M} \times (K^{\circ\circ})^{n+N} \to (K^\circ)^m \times (K^{\circ\circ})^n.$$

If the set of points in $\pi(X)$ with d-dimensional fibers is somewhere dense, then

$$\text{g-dim } X \geq m + n + d.$$



10 Further Results

-Additionally the following theorems hold if Char $K = 0$.

Theorem: Let $X \subset (K^{\circ})^{m+M} \times (K^{\circ\circ})^{n+N}$ be a D-semianalytic set, then there are finitely many D-semianalytic manifolds X_i such that

i) $X = \bigcup_i X_i$

ii) for each $\bar{p} \in (K^{\circ})^m \times (K^{\circ\circ})^n$, the fiber set $X_{\bar{p}}$ is either empty or a D-semianalytic manifold.

Theorem: If X is a *somewhere dense subanalytic set*, then X contains an interior point.

Corollary: For a subanalytic set X,

$$\text{g-dim } X = \text{w-dim } X.$$

-Bartenwerfer introduced *piece number of* a D-semianalytic set to measure its complexity.

Theorem: For a D-semianalytic set $X \subset (K^{\circ})^{m+M} \times (K^{\circ\circ})^{n+N}$, there is a bound α such that for all $\bar{p} \in (K^{\circ})^m \times (K^{\circ\circ})^n$, the piece number of each of the fibers $X_{\bar{p}}$ is bounded by α.