Suggested Homework Problems #8

MATH 516-01 Fall 2001

Please do **all** of the following exercises:

• Given a set X. Show that:

(a) If X is given the *trivial topology*, then X is *compact*.

(b) If X is given the *discrete topology*, then X is *compact* if and only if X is finite.

• Show that every *compact* subspace of a metric space is closed and bounded in that metric. Find a metric space in which not every closed and bounded subspace is compact.

• Let $\{U_\alpha\}$ be an open cover of a compact metric space X. Show that there exists $\epsilon > 0$ such that for any $x \in X$, there exists $\alpha(x)$ such that $B_\epsilon(x) \subset U_{\alpha(x)}$, where $B_\epsilon(x)$ is the ϵ-ball centered at x in the metric space X.

• Let $\mathbf{O}(n)$ be the set of all $n \times n$ orthogonal real matrices. Show that:

(a) $\mathbf{O}(n)$ is *compact*

(b) $\mathbf{O}(n)$ is not connected. [Remark: It turns out $\mathbf{O}(n)$ has precisely two *arc components*, one of which is the special orthogonal group $\mathbf{SO}(n)$]